

\section*{Historical Background}

In this section, we will briefly discuss historical developments that will help put the incompleteness theorems in context. In particular, we will give a very sketchy overview of the history of mathematical logic; and then say a few words about the history of the foundations of mathematics.

The phrase “mathematical logic” is ambiguous. One can interpret the word “mathematical” as describing the subject matter, as in, “the logic of mathematics,” denoting the principles of mathematical reasoning; or as describing the methods, as in “the mathematics of logic,” denoting a mathematical study of the principles of reasoning. The account that follows involves mathematical logic in both senses, often at the same time.

The study of logic began, essentially, with Aristotle, who lived approximately 384–322 BCE. His \textit{Categories}, \textit{Prior analytics}, and \textit{Posterior analytics} include systematic studies of the principles of scientific reasoning, including a thorough and systematic study of the syllogism.

Aristotle’s logic dominated scholastic philosophy through the middle ages; indeed, as late as the eighteenth century, Kant maintained that Aristotle’s logic was perfect and in no need of revision. But the theory of the syllogism is far too limited to model anything but the most superficial aspects of mathematical reasoning. A century earlier, Leibniz, a contemporary of Newton’s, imagined a complete “calculus” for logical reasoning, and made some rudimentary steps towards designing such a calculus, essentially describing a version of propositional logic.

The nineteenth century was a watershed for logic. In 1854 George Boole wrote \textit{The Laws of Thought}, with a thorough algebraic study of propositional logic that is not far from modern presentations. In 1879 Gottlob Frege published his \textit{Begriffsschrift} (Concept writing) which extends propositional logic with quantifiers and relations, and thus includes first-order logic. In fact, Frege’s logical systems included higher-order logic as well, and more. In his \textit{Basic Laws of Arithmetic}, Frege set out to show that all of arithmetic could be derived in his Begriffsschrift from purely logical assumption. Unfortunately, these assumptions turned out to be inconsistent, as Russell showed in 1902. But setting aside the inconsistent axiom, Frege more or less invented modern logic singlehandedly, a startling achievement. Quantificational logic was also developed independently by algebraically-minded thinkers after Boole, including Peirce and Schröder.

Let us now turn to developments in the foundations of mathematics. Of course, since logic plays an important role in mathematics, there is a good deal of interaction with the developments just described. For example, Frege developed his logic with the explicit purpose of showing that all of mathematics could be based solely on his logical framework; in particular, he wished to show that mathematics consists of a priori analytic truths instead of, as Kant had maintained, a priori synthetic ones.

Many take the birth of mathematics proper to have occurred with the Greeks. Euclid’s \textit{Elements}, written around 300 B.C., is already a mature repre-
sentative of Greek mathematics, with its emphasis on rigor and precision. The
definitions and proofs in Euclid’s *Elements* survive more or less intact in high
school geometry textbooks today (to the extent that geometry is still taught
in high schools). This model of mathematical reasoning has been held to be a
paradigm for rigorous argumentation not only in mathematics but in branches
of philosophy as well. (Spinoza even presented moral and religious arguments
in the Euclidean style, which is strange to see!)

Calculus was invented by Newton and Leibniz in the seventeenth century.
(A fierce priority dispute raged for centuries, but most scholars today hold that
the two developments were for the most part independent.) Calculus involves
reasoning about, for example, infinite sums of infinitely small quantities; these
features fueled criticism by Bishop Berkeley, who argued that belief in God was
no less rational than the mathematics of his time. The methods of calculus
were widely used in the eighteenth century, for example by Leonhard Euler,
who used calculations involving infinite sums with dramatic results.

In the nineteenth century, mathematicians tried to address Berkeley’s crit-
icisms by putting calculus on a firmer foundation. Efforts by Cauchy, Weier-
strass, Bolzano, and others led to our contemporary definitions of limits, con-
tinuity, differentiation, and integration in terms of “epslons and deltas,” in other
words, devoid of any reference to infinitesimals. Later in the century, mathe-
maticians tried to push further, and explain all aspects of calculus, including
the real numbers themselves, in terms of the natural numbers. (Kronecker:
“God created the whole numbers, all else is the work of man.”) In 1872,
Dedekind wrote “Continuity and the irrational numbers,” where he showed
how to “construct” the real numbers as sets of rational numbers (which, as
you know, can be viewed as pairs of natural numbers); in 1888 he wrote “Was
sind und was sollen die Zahlen” (roughly, “What are the natural numbers, and
what should they be?”) which aimed to explain the natural numbers in purely
“logical” terms. In 1887 Kronecker wrote “Über den Zahlbegriff” (“On the
concept of number”) where he spoke of representing all mathematical object
in terms of the integers; in 1889 Giuseppe Peano gave formal, symbolic axioms
for the natural numbers.

The end of the nineteenth century also brought a new boldness in dealing
with the infinite. Before then, infinitary objects and structures (like the set
of natural numbers) were treated gingerly; “infinitely many” was understood
as “as many as you want,” and “approaches in the limit” was understood as
“gets as close as you want.” But Georg Cantor showed that it was possible to
take the infinite at face value. Work by Cantor, Dedekind, and others help to
introduce the general set-theoretic understanding of mathematics that is now
widely accepted.

This brings us to twentieth century developments in logic and foundations.
In 1902 Russell discovered the paradox in Frege’s logical system. In 1904 Zer-
melo proved Cantor’s well-ordering principle, using the so-called “axiom of
choice”; the legitimacy of this axiom prompted a good deal of debate. Between
1910 and 1913 the three volumes of Russell and Whitehead’s *Principia Mathe-
natica* appeared, extending the Fregean program of establishing mathematics

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on logical grounds. Unfortunately, Russell and Whitehead were forced to adopt
two principles that seemed hard to justify as purely logical: an axiom of in-
finity and an axiom of “reducibility.” In the 1900’s Poincaré criticized the use
of “impredicative definitions” in mathematics, and in the 1910’s Brouwer be-
gan proposing to refound all of mathematics in an “intuitionistic” basis, which
avoided the use of the law of the excluded middle ($\varphi \lor \neg \varphi$).
Strange days indeed! The program of reducing all of mathematics to logic
is now referred to as “logicism,” and is commonly viewed as having failed, due
to the difficulties mentioned above. The program of developing mathematics
in terms of intuitionistic mental constructions is called “intuitionism,” and is
viewed as posing overly severe restrictions on everyday mathematics. Around
the turn of the century, David Hilbert, one of the most influential mathem-
aticians of all time, was a strong supporter of the new, abstract methods
introduced by Cantor and Dedekind: “no one will drive us from the paradise
that Cantor has created for us.” At the same time, he was sensitive to founda-
tional criticisms of these new methods (oddly enough, now called “classical”).
He proposed a way of having one’s cake and eating it too:

1. Represent classical methods with formal axioms and rules; represent
   mathematical questions as formulas in an axiomatic system.

2. Use safe, “finitary” methods to prove that these formal deductive systems
   are consistent.

Hilbert’s work went a long way toward accomplishing the first goal. In 1899,
he had done this for geometry in his celebrated book *Foundations of geometry*.
In subsequent years, he and a number of his students and collaborators worked
on other areas of mathematics to do what Hilbert had done for geometry.
Hilbert himself gave axiom systems for arithmetic and analysis. Zermelo gave
an axiomatization of set theory, which was expanded on by Fraenkel, Skolem,
von Neumann, and others. By the mid-1920s, there were two approaches that
laid claim to the title of an axiomatization of “all” of mathematics, the
*Principia mathematica* of Russell and Whitehead, and what came to be known as
Zermelo-Fraenkel set theory.

In 1921, Hilbert set out on a research project to establish the goal of proving
these systems to be consistent. He was aided in this project by several of his
students, in particular Bernays, Ackermann, and later Gentzen. The basic
idea for accomplishing this goal was to cast the question of the possibility of
a derivation of an inconsistency in mathematics as a combinatorial problem
about possible sequences of symbols, namely possible sequences of sentences
which meet the criterion of being a correct derivation of, say, $\varphi \land \neg \varphi$
from the axioms of an axiom system for arithmetic, analysis, or set theory. A proof
of the impossibility of such a sequence of symbols would—since it is itself
a mathematical proof—be formalizable in these axiomatic systems. In other
words, there would be some sentence $\text{Con}$ which states that, say, arithmetic
is consistent. Moreover, this sentence should be provable in the systems in
question, especially if its proof requires only very restricted, “finitary” means.
The second aim, that the axiom systems developed would settle every mathematical question, can be made precise in two ways. In one way, we can formulate it as follows: For any sentence \( \varphi \) in the language of an axiom system for mathematics, either \( \varphi \) or \( \neg \varphi \) is provable from the axioms. If this were true, then there would be no sentences which can neither be proved nor refuted on the basis of the axioms, no questions which the axioms do not settle. An axiom system with this property is called complete. Of course, for any given sentence it might still be a difficult task to determine which of the two alternatives holds. But in principle there should be a method to do so. In fact, for the axiom and derivation systems considered by Hilbert, completeness would imply that such a method exists—although Hilbert did not realize this. The second way to interpret the question would be this stronger requirement: that there be a mechanical, computational method which would determine, for a given sentence \( \varphi \), whether it is derivable from the axioms or not.

In 1931, Gödel proved the two “incompleteness theorems,” which showed that this program could not succeed. There is no axiom system for mathematics which is complete, specifically, the sentence that expresses the consistency of the axioms is a sentence which can neither be proved nor refuted.

This struck a lethal blow to Hilbert’s original program. However, as is so often the case in mathematics, it also opened up exciting new avenues for research. If there is no one, all-encompassing formal system of mathematics, it makes sense to develop more circumscribed systems and investigate what can be proved in them. It also makes sense to develop less restricted methods of proof for establishing the consistency of these systems, and to find ways to measure how hard it is to prove their consistency. Since Gödel showed that (almost) every formal system has questions it cannot settle, it makes sense to look for “interesting” questions a given formal system cannot settle, and to figure out how strong a formal system has to be to settle them. To the present day, logicians have been pursuing these questions in a new mathematical discipline, the theory of proofs.

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**Bibliography**