Can we modify Gödel’s proof to get a stronger result, replacing “ω-consistent” with simply “consistent”? The answer is “yes,” using a trick discovered by Rosser. Rosser’s trick is to use a “modified” derivability predicate $RProv_T(y)$ instead of $Prov_T(y)$.

**Theorem inp.1.** Let $T$ be any consistent, axiomatizable theory extending $Q$. Then $T$ is not complete.

*Proof.* Recall that $Prov_T(y)$ is defined as $\exists x Prf_T(x, y)$, where $Prf_T(x, y)$ represents the decidable relation which holds iff $x$ is the Gödel number of a derivation of the sentence with Gödel number $y$. The relation that holds between $x$ and $y$ if $x$ is the Gödel number of a refutation of the sentence with Gödel number $y$ is also decidable. Let $not(x)$ be the primitive recursive function which does the following: if $x$ is the code of a formula $\varphi$, $not(x)$ is a code of $\neg \varphi$. Then $Ref_T(x, y)$ holds iff $Prf_T(x, not(y))$. Let $Ref_T(x, y)$ represent it. Then, if $T \vdash \neg \varphi$ and $\delta$ is a corresponding derivation, $Q \vdash Ref_T(\delta, \varphi)$. We define $RProv_T(y)$ as $\exists x (Prf_T(x, y) \land \forall z (z < x \rightarrow \neg Ref_T(z, y)))$.

Roughly, $RProv_T(y)$ says “there is a proof of $y$ in $T$, and there is no shorter refutation of $y$.” Assuming $T$ is consistent, $RProv_T(y)$ is true of the same numbers as $Prov_T(y)$; but from the point of view of provability in $T$ (and we now know that there is a difference between truth and provability!) the two have different properties. If $T$ is inconsistent, then the two do not hold of the same numbers! ($RProv_T(y)$ is often read as “$y$ is Rosser provable.” Since, as just discussed, Rosser provability is not some special kind of provability—in inconsistent theories, there are sentences that are provable but not Rosser provable—this may be confusing. To avoid the confusion, you could instead read it as “$y$ is shmovable.”)

By the fixed-point lemma, there is a formula $\rho_T$ such that

$$Q \vdash \rho_T \leftrightarrow \neg RProv_T(\rho_T).$$

In contrast to the proof of ??, here we claim that if $T$ is consistent, $T$ doesn’t derive $\rho_T$, and $T$ also doesn’t derive $\neg \rho_T$. (In other words, we don’t need the assumption of ω-consistency.)

First, let’s show that $T \nvdash \rho_T$. Suppose it did, so there is a derivation of $\rho_T$ from $T$; let $n$ be its Gödel number. Then $Q \vdash Prf_T(\pi, \rho_T)$, since $Prf_T$ represents $Prf_T$ in $Q$. Also, for each $k < n$, $k$ is not the Gödel number of a derivation of $\neg \rho_T$, since $T$ is consistent. So for each $k < n$, $Q \vdash \neg Ref_T(\delta, \rho_T)$. By ??, $Q \vdash \exists z (z < \pi \rightarrow \neg Ref_T(z, \rho_T))$. Thus,

$$Q \vdash \exists x (Prf_T(x, \rho_T) \land \forall z (z < x \rightarrow \neg Ref_T(z, \rho_T))).$$
but that’s just $\text{RProv}_T(\rho_T \neg \gamma)$. By eq. (1), $Q \vdash \neg \rho_T$. Since $T$ extends $Q$, also $T \vdash \neg \rho_T$. We’ve assumed that $T \vdash \rho_T$, so $T$ would be inconsistent, contrary to the assumption of the theorem.

Now, let’s show that $T \not\vdash \neg \rho_T$. Again, suppose it did, and suppose $n$ is the Gödel number of a derivation of $\neg \rho_T$. Then $\text{Ref}_T(n, \#_\rho_T \#)$ holds, and since $\text{Ref}_T$ represents $\text{Ref}_T$ in $Q$, $Q \vdash \text{Ref}_T(\pi, \rho_T \neg \gamma)$. We’ll again show that $T$ would then be inconsistent because it would also derive $\rho_T$. Since $Q \vdash \rho_T \leftrightarrow \neg \text{RProv}_T(\rho_T \neg \gamma)$, and since $T$ extends $Q$, it suffices to show that $Q \vdash \neg \text{RProv}_T(\rho_T \neg \gamma)$.

The sentence $\neg \text{RProv}_T(\rho_T \neg \gamma)$, i.e.,

\[
\neg \exists x \left( \text{Prf}_T(x, \rho_T \neg \gamma) \land \forall z \left( z < x \rightarrow \neg \text{Ref}_T(z, \rho_T \neg \gamma) \right) \right),
\]

is logically equivalent to

\[
\forall x \left( \text{Prf}_T(x, \rho_T \neg \gamma) \rightarrow \exists z \left( z < x \land \text{Ref}_T(z, \rho_T \neg \gamma) \right) \right).
\]

We argue informally using logic, making use of facts about what $Q$ derives. Suppose $x$ is arbitrary and $\text{Prf}_T(x, \rho_T \neg \gamma)$. We already know that $T \not\vdash \rho_T$, and so for every $k$, $Q \vdash \neg \text{Prf}_T(k, \rho_T \neg \gamma)$. Thus, for every $k$ it follows that $x \neq k$. In particular, we have (a) that $x \neq \pi$. We also have $\neg(x = \emptyset \lor x = 1 \lor \cdots \lor x = n-1)$ and so by (a), $\neg(x < \pi)$. By (a), $x < \pi$. Since $Q \vdash \text{Ref}_T(\pi, \rho_T \neg \gamma)$, we have $\pi < x \land \text{Ref}_T(\pi, \rho_T \neg \gamma)$, and from that $\exists z \left( z < x \land \text{Ref}_T(z, \rho_T \neg \gamma) \right)$. Since $x$ was arbitrary we get, as required, that

\[
\forall x \left( \text{Prf}_T(x, \rho_T \neg \gamma) \rightarrow \exists z \left( z < x \land \text{Ref}_T(z, \rho_T \neg \gamma) \right) \right). \quad \square
\]

**Problem inp.1.** Two sets $A$ and $B$ of natural numbers are said to be **computably inseparable** if there is no decidable set $X$ such that $A \subseteq X$ and $B \subseteq \overline{X}$ ($\overline{X}$ is the complement, $\mathbb{N} \setminus X$, of $X$). Let $T$ be a consistent axiomatizable extension of $Q$. Suppose $A$ is the set of Gödel numbers of sentences provable in $T$ and $B$ the set of Gödel numbers of sentences refutable in $T$. Prove that $A$ and $B$ are computably inseparable.

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**Bibliography**