

inp.1 The Provability Conditions for PA

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sec Peano arithmetic, or **PA**, is the theory extending **Q** with induction axioms for all **formulas**. In other words, one adds to **Q** axioms of the form

$$(\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x \varphi(x)$$

for every **formula** φ . Notice that this is really a *schema*, which is to say, infinitely many axioms (and it turns out that **PA** is *not* finitely axiomatizable). But since one can effectively determine whether or not a string of symbols is an instance of an induction axiom, the set of axioms for **PA** is computable. **PA** is a much more robust theory than **Q**. For example, one can easily prove that addition and multiplication are commutative, using induction in the usual way. In fact, most finitary number-theoretic and combinatorial arguments can be carried out in **PA**.

Since **PA** is computably axiomatized, the provability predicate $\text{Prf}_{\mathbf{PA}}(x, y)$ is computable and hence represented in **Q** (and so, in **PA**). As before, I will take $\text{Prf}_{\mathbf{PA}}(x, y)$ to denote the formula representing the relation. Let $\text{Prov}_{\mathbf{PA}}(y)$ be the formula $\exists x \text{Prf}_{\mathbf{PA}}(x, y)$, which, intuitively says, “ y is provable from the axioms of **PA**.” The reason we need a little bit more than the axioms of **Q** is we need to know that the theory we are using is strong enough to prove a few basic facts about this provability predicate. In fact, what we need are the following facts:

P1. If $\mathbf{PA} \vdash \varphi$, then $\mathbf{PA} \vdash \text{Prov}_{\mathbf{PA}}(\ulcorner \varphi \urcorner)$

P2. For all **formulas** φ and ψ ,

$$\mathbf{PA} \vdash \text{Prov}_{\mathbf{PA}}(\ulcorner \varphi \rightarrow \psi \urcorner) \rightarrow (\text{Prov}_{\mathbf{PA}}(\ulcorner \varphi \urcorner) \rightarrow \text{Prov}_{\mathbf{PA}}(\ulcorner \psi \urcorner))$$

P3. For every **formula** φ ,

$$\mathbf{PA} \vdash \text{Prov}_{\mathbf{PA}}(\ulcorner \varphi \urcorner) \rightarrow \text{Prov}_{\mathbf{PA}}(\ulcorner \text{Prov}_{\mathbf{PA}}(\ulcorner \varphi \urcorner) \urcorner).$$

The only way to verify that these three properties hold is to describe the **formula** $\text{Prov}_{\mathbf{PA}}(y)$ carefully and use the axioms of **PA** to describe the relevant formal proofs. Conditions (1) and (2) are easy; it is really condition (3) that requires work. (Think about what kind of work it entails. . .) Carrying out the details would be tedious and uninteresting, so here we will ask you to take it on faith that **PA** has the three properties listed above. A reasonable choice of $\text{Prov}_{\mathbf{PA}}(y)$ will also satisfy

P4. If $\mathbf{PA} \vdash \text{Prov}_{\mathbf{PA}}(\ulcorner \varphi \urcorner)$, then $\mathbf{PA} \vdash \varphi$.

But we will not need this fact.

Incidentally, Gödel was lazy in the same way we are being now. At the end of the 1931 paper, he sketches the proof of the second incompleteness theorem, and promises the details in a later paper. He never got around to it; since digression

everyone who understood the argument believed that it could be carried out
(he did not need to fill in the details.)

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Bibliography