It is worthwhile to spend some time with Gödel’s 1931 paper. The introduction sketches the ideas we have just discussed. Even if you just skim through the paper, it is easy to see what is going on at each stage: first Gödel describes the formal system $P$ (syntax, axioms, proof rules); then he defines the primitive recursive functions and relations; then he shows that $xBy$ is primitive recursive, and argues that the primitive recursive functions and relations are represented in $P$. He then goes on to prove the incompleteness theorem, as above. In Section 3, he shows that one can take the unprovable assertion to be a sentence in the language of arithmetic. This is the origin of the $\beta$-lemma, which is what we also used to handle sequences in showing that the recursive functions are representable in $Q$. Gödel doesn’t go so far to isolate a minimal set of axioms that suffice, but we now know that $Q$ will do the trick. Finally, in Section 4, he sketches a proof of the second incompleteness theorem.