

inp.1 The First Incompleteness Theorem

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We can now describe Gödel’s original proof of the first incompleteness theorem. Let \mathbf{T} be any computably axiomatized theory in a language extending the language of arithmetic, such that \mathbf{T} includes the axioms of \mathbf{Q} . This means that, in particular, \mathbf{T} represents computable functions and relations.

We have argued that, given a reasonable coding of formulas and proofs as numbers, the relation $\text{Prf}_T(x, y)$ is computable, where $\text{Prf}_T(x, y)$ holds if and only if x is the Gödel number of a **derivation** of the **formula** with Gödel number y in \mathbf{T} . In fact, for the particular theory that Gödel had in mind, Gödel was able to show that this relation is primitive recursive, using the list of 45 functions and relations in his paper. The 45th relation, xBy , is just $\text{Prf}_T(x, y)$ for his particular choice of \mathbf{T} . Remember that where Gödel uses the word “recursive” in his paper, we would now use the phrase “primitive recursive.”

Since $\text{Prf}_T(x, y)$ is computable, it is representable in \mathbf{T} . We will use $\text{Prf}_T(x, y)$ to refer to the formula that represents it. Let $\text{Prov}_T(y)$ be the formula $\exists x \text{Prf}_T(x, y)$. This describes the 46th relation, $\text{Bew}(y)$, on Gödel’s list. As Gödel notes, this is the only relation that “cannot be asserted to be recursive.” What he probably meant is this: from the definition, it is not clear that it is computable; and later developments, in fact, show that it isn’t.

Let \mathbf{T} be an **axiomatizable** theory containing \mathbf{Q} . Then $\text{Prf}_T(x, y)$ is decidable, hence representable in \mathbf{Q} by a **formula** $\text{Prf}_T(x, y)$. Let $\text{Prov}_T(y)$ be the formula we described above. By the fixed-point lemma, there is a formula $\gamma_{\mathbf{T}}$ such that \mathbf{Q} (and hence \mathbf{T}) **derives**

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$$\gamma_{\mathbf{T}} \leftrightarrow \neg \text{Prov}_T(\ulcorner \gamma_{\mathbf{T}} \urcorner). \quad (1)$$

Note that $\gamma_{\mathbf{T}}$ says, in essence, “ $\gamma_{\mathbf{T}}$ is not **derivable** in \mathbf{T} .”

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lem:cons-G-unprov

Lemma inp.1. *If \mathbf{T} is a consistent, **axiomatizable** theory extending \mathbf{Q} , then $\mathbf{T} \not\vdash \gamma_{\mathbf{T}}$.*

Proof. Suppose \mathbf{T} **derives** $\gamma_{\mathbf{T}}$. Then there is a **derivation**, and so, for some number m , the relation $\text{Prf}_T(m, \# \gamma_{\mathbf{T}} \#)$ holds. But then \mathbf{Q} **derives** the sentence $\text{Prf}_T(\bar{m}, \ulcorner \gamma_{\mathbf{T}} \urcorner)$. So \mathbf{Q} **derives** $\exists x \text{Prf}_T(x, \ulcorner \gamma_{\mathbf{T}} \urcorner)$, which is, by definition, $\text{Prov}_T(\ulcorner \gamma_{\mathbf{T}} \urcorner)$. By **eq. (1)**, \mathbf{Q} **derives** $\neg \gamma_{\mathbf{T}}$, and since \mathbf{T} extends \mathbf{Q} , so does \mathbf{T} . We have shown that if \mathbf{T} **derives** $\gamma_{\mathbf{T}}$, then it also **derives** $\neg \gamma_{\mathbf{T}}$, and hence it would be inconsistent. \square

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Definition inp.2. A theory \mathbf{T} is ω -consistent if the following holds: if $\exists x \varphi(x)$ is any sentence and \mathbf{T} **derives** $\neg \varphi(\bar{0}), \neg \varphi(\bar{1}), \neg \varphi(\bar{2}), \dots$ then \mathbf{T} does not prove $\exists x \varphi(x)$.

Note that every ω -consistent theory is also consistent. This follows simply from the fact that if \mathbf{T} is inconsistent, then $\mathbf{T} \vdash \varphi$ for every φ . In particular, if \mathbf{T} is inconsistent, it **derives** both $\neg \varphi(\bar{n})$ for every n and also **derives** $\exists x \varphi(x)$. So, if \mathbf{T} is inconsistent, it is ω -inconsistent. By contraposition, if \mathbf{T} is ω -consistent, it must be consistent.

Lemma inp.3. *If \mathbf{T} is an ω -consistent, axiomatizable theory extending \mathbf{Q} , then $\mathbf{T} \not\vdash \neg\gamma_{\mathbf{T}}$.* [inc:inp:1in:](#)
[lem:omega-cons-G-unref](#)

Proof. We show that if \mathbf{T} derives $\neg\gamma_{\mathbf{T}}$, then it is ω -inconsistent. Suppose \mathbf{T} derives $\neg\gamma_{\mathbf{T}}$. If \mathbf{T} is inconsistent, it is ω -inconsistent, and we are done. Otherwise, \mathbf{T} is consistent, so it does not derive $\gamma_{\mathbf{T}}$ by Lemma inp.1. Since there is no derivation of $\gamma_{\mathbf{T}}$ in \mathbf{T} , \mathbf{Q} derives

$$\neg\text{Prf}_T(\bar{0}, \ulcorner \gamma_{\mathbf{T}} \urcorner), \neg\text{Prf}_T(\bar{1}, \ulcorner \gamma_{\mathbf{T}} \urcorner), \neg\text{Prf}_T(\bar{2}, \ulcorner \gamma_{\mathbf{T}} \urcorner), \dots$$

and so does \mathbf{T} . On the other hand, by eq. (1), $\neg\gamma_{\mathbf{T}}$ is equivalent to $\exists x \text{Prf}_T(x, \ulcorner \gamma_{\mathbf{T}} \urcorner)$. So \mathbf{T} is ω -inconsistent. □

Problem inp.1. Every ω -consistent theory is consistent. Show that the converse does not hold, i.e., that there are consistent but ω -inconsistent theories. Do this by showing that $\mathbf{Q} \cup \{\neg\gamma_{\mathbf{Q}}\}$ is consistent but ω -inconsistent.

Theorem inp.4. *Let \mathbf{T} be any ω -consistent, axiomatizable theory extending \mathbf{Q} . Then \mathbf{T} is not complete.* [inc:inp:1in:](#)
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Proof. If \mathbf{T} is ω -consistent, it is consistent, so $\mathbf{T} \not\vdash \gamma_{\mathbf{T}}$ by Lemma inp.1. By Lemma inp.3, $\mathbf{T} \not\vdash \neg\gamma_{\mathbf{T}}$. This means that \mathbf{T} is incomplete, since it derives neither $\gamma_{\mathbf{T}}$ nor $\neg\gamma_{\mathbf{T}}$. □

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Bibliography