inp.1 The First Incompleteness Theorem

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We can now describe Gödel's original proof of the first incompleteness theorem. Let $\mathbf{T}$ be any computably axiomatized theory in a language extending the language of arithmetic, such that $\mathbf{T}$ includes the axioms of $\mathbf{Q}$. This means that, in particular, $\mathbf{T}$ represents computable functions and relations.

We have argued that, given a reasonable coding of formulas and proofs as numbers, the relation $\mathrm{Prf}_T(x,y)$ is computable, where $\mathrm{Prf}_T(x,y)$ holds if and only if x is the Gödel number of a derivation of the formula with Gödel number y in $\mathbf{T}$. In fact, for the particular theory that Gödel had in mind, Gödel was able to show that this relation is primitive recursive, using the list of 45 functions and relations in his paper. The 45th relation, xBy, is just $\mathrm{Prf}_T(x,y)$ for his particular choice of $\mathbf{T}$. Remember that where Gödel uses the word "recursive" in his paper, we would now use the phrase "primitive recursive."

Since $\mathrm{Prf}_T(x,y)$ is computable, it is representable in $\mathbf{T}$. We will use $\mathsf{Prf}_T(x,y)$ to refer to the formula that represents it. Let $\mathsf{Prov}_T(y)$ be the formula $\exists x\,\mathsf{Prf}_T(x,y)$. This describes the 46th relation, $\mathrm{Bew}(y)$, on Gödel's list. As Gödel notes, this is the only relation that "cannot be asserted to be recursive." What he probably meant is this: from the definition, it is not clear that it is computable; and later developments, in fact, show that it isn't.

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Definition inp.1. A theory $\mathbf{T}$ is ω-consistent if the following holds: if $\exists x\,\varphi(x)$ is any sentence and $\mathbf{T}$ proves $\lnot\varphi(\overline{0})$, $\lnot\varphi(\overline{1})$, $\lnot\varphi(\overline{2})$, ... then $\mathbf{T}$ does not prove $\exists x\,\varphi(x)$.

We can now prove the following.

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Theorem inp.2. *Let $\mathbf{T}$ be any ω-consistent, axiomatizable theory extending $\mathbf{Q}$. Then $\mathbf{T}$ is not complete.*

Proof. Let $\mathbf{T}$ be an axiomatizable theory containing $\mathbf{Q}$ Then $\mathrm{Prf}_T(x,y)$ is decidable, hence representable in $\mathbf{Q}$ by a formula $\mathsf{Prf}_T(x,y)$. Let $\mathsf{Prov}_T(y)$ be the formula we described above. By the fixed-point lemma, there is a formula $\gamma_{\mathbf{T}}$ such that $\mathbf{Q}$ (and hence $\mathbf{T}$) proves

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$$\gamma_{\mathbf{T}} \leftrightarrow \lnot\mathsf{Prov}_T(\ulcorner\gamma_{\mathbf{T}}\urcorner). \tag{1}$$

Note that φ says, in essence, "φ is not provable."

We claim that

1. If $\mathbf{T}$ is consistent, $\mathbf{T}$ doesn't prove $\gamma_{\mathbf{T}}$
2. If $\mathbf{T}$ is ω-consistent, $\mathbf{T}$ doesn't prove $\lnot\gamma_{\mathbf{T}}$.

This means that if $\mathbf{T}$ is ω-consistent, it is incomplete, since it proves neither $\gamma_{\mathbf{T}}$ nor $\lnot\gamma_{\mathbf{T}}$. Let us take each claim in turn.

Suppose $\mathbf{T}$ proves $\gamma_{\mathbf{T}}$. Then there *is* a derivation, and so, for some number m, the relation $\mathrm{Prf}_T(m, {}^{\#}\gamma_{\mathbf{T}}{}^{\#})$ holds. But then $\mathbf{Q}$ proves the sentence $\mathsf{Prf}_T(\overline{m},\ulcorner\gamma_{\mathbf{T}}\urcorner)$. So $\mathbf{Q}$ proves $\exists x\,\mathsf{Prf}_T(x,\ulcorner\gamma_{\mathbf{T}}\urcorner)$, which is, by definition, $\mathsf{Prov}_T(\ulcorner\gamma_{\mathbf{T}}\urcorner)$.

By eq. (1), $\mathbf{Q}$ proves $\neg\gamma_{\mathbf{T}}$, and since $\mathbf{T}$ extends $\mathbf{Q}$, so does $\mathbf{T}$. We have shown that if $\mathbf{T}$ proves $\gamma_{\mathbf{T}}$, then it also proves $\neg\gamma_{\mathbf{T}}$, and hence it would be inconsistent.

For the second claim, let us show that if $\mathbf{T}$ proves $\neg\gamma_{\mathbf{T}}$, then it is ω-inconsistent. Suppose $\mathbf{T}$ proves $\neg\gamma_{\mathbf{T}}$. If $\mathbf{T}$ is inconsistent, it is ω-inconsistent, and we are done. Otherwise, $\mathbf{T}$ is consistent, so it does not prove $\gamma_{\mathbf{T}}$. Since there is no proof of $\gamma_{\mathbf{T}}$ in $\mathbf{T}$, $\mathbf{Q}$ proves

$$\neg\mathsf{Prf}_T(\overline{0}, \ulcorner\gamma_{\mathbf{T}}\urcorner), \neg\mathsf{Prf}_T(\overline{1}, \ulcorner\gamma_{\mathbf{T}}\urcorner), \neg\mathsf{Prf}_T(\overline{2}, \ulcorner\gamma_{\mathbf{T}}\urcorner), \ldots$$

and so does $\mathbf{T}$. On the other hand, by eq. (1), $\neg\gamma_{\mathbf{T}}$ is equivalent to $\exists x\, \mathsf{Prf}_T(x, \ulcorner\gamma_{\mathbf{T}}\urcorner)$. So $\mathbf{T}$ is ω-inconsistent. □

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Bibliography