art.1 Derivations in Natural Deduction

inc:art:pnd:sec In order to arithmetize derivations, we must represent derivations as numbers. Since derivations are trees of formulas where each inference carries one or two labels, a recursive representation is the most obvious approach: we represent a derivation as a tuple, the components of which are the end-formula, the labels, and the representations of the sub-derivations leading to the premises of the last inference. explanation

Definition art.1. If δ is a derivation in natural deduction, then ${}^{\#}\delta^{\#}$ is

1. $\langle 0, {}^{\#}\varphi^{\#}, n\rangle$ if δ consists only of the assumption φ. The number n is 0 if it is an undischarged assumption, and the numerical label otherwise.

2. $\langle 1, {}^{\#}\varphi^{\#}, n, k, {}^{\#}\delta_1{}^{\#}\rangle$ if δ ends in an inference with one premise, k is given by the following table according to which rule was used in the last inference, and δ_1 is the immediate subproof ending in the premise of the last inference. n is the label of the inference, or 0 if the inference does not discharge any assumptions.

Rule:	$\wedge$Elim	$\vee$Intro	$\rightarrow$Intro	$\neg$Intro	$\bot_I$
k:	1	2	3	4	5

Rule:	$\bot_C$	$\forall$Intro	$\forall$Elim	$\exists$Intro	$=$Intro
k:	6	7	8	9	10

3. $\langle 2, {}^{\#}\varphi^{\#}, n, k, {}^{\#}\delta_1{}^{\#}, {}^{\#}\delta_2{}^{\#}\rangle$ if δ ends in an inference with two premises, k is given by the following table according to which rule was used in the last inference, and δ_1, δ_2 are the immediate subderivations ending in the left and right premise of the last inference, respectively. n is the label of the inference, or 0 if the inference does not discharge any assumptions.

Rule:	$\wedge$Intro	$\rightarrow$Elim	$\neg$Elim
k:	1	2	3

4. $\langle 3, {}^{\#}\varphi^{\#}, n, {}^{\#}\delta_1{}^{\#}, {}^{\#}\delta_2{}^{\#}, {}^{\#}\delta_3{}^{\#}\rangle$ if δ ends in an $\vee$Elim inference. δ_1, δ_2, δ_3 are the immediate subderivations ending in the left, middle, and right premise of the last inference, respectively, and n is the label of the inference.

Example art.2. Consider the very simple derivation

$$\cfrac{\cfrac{[(\varphi \wedge \psi)]^1}{\varphi}\,\wedge\text{Elim}}{(\varphi \rightarrow \psi)}\,{}_1\rightarrow\text{Intro}$$

The Gödel number of the assumption would be $d_0 = \langle 0, {}^{\#}(\varphi \wedge \psi)^{\#}, 1\rangle$. The Gödel number of the derivation ending in the conclusion of $\wedge$Elim would be

$d_1 = \langle 1, {}^{\#}\varphi^{\#}, 0, 1, d_0\rangle$ (1 since $\wedge$Elim has one premise, Gödel number of conclusion φ, 0 because no assumption is discharged, 1 is the number coding $\wedge$Elim). The Gödel number of the entire derivation then is $\langle 1, {}^{\#}(\varphi \to \psi)^{\#}, 1, 3, d_1\rangle$, i.e.,

$$2^2 \cdot 3^{{}^{\#}(\varphi\to\psi)^{\#}+1} \cdot 5^2 \cdot 7^4 \cdot 11^{(2^2 \cdot 3^{{}^{\#}\varphi^{\#}+1} \cdot 5^1 \cdot 7^2 \cdot 11^{(2^1 \cdot 3^{{}^{\#}(\varphi\wedge\psi)^{\#}+1} \cdot 5^2)})}.$$

explanation

Having settled on a representation of derivations, we must also show that we can manipulate such derivations primitive recursively, and express their essential properties and relations so. Some operations are simple: e.g., given a Gödel number d of a derivation, $(d)_1$ gives us the Gödel number of its end-formula. Some are much harder. We'll at least sketch how to do this. The goal is to show that the relation "δ is a derivation of φ from Γ" is primitive recursive on the Gödel numbers of δ and φ.

Proposition art.3. *The following relations are primitive recursive:* *inc:art:pnd: prop:followsby*

1. *φ occurs as an assumption in δ with label n.*
2. *All assumption in δ with label n are of the form φ (i.e., we can discharge the assumption φ using label n in δ).*
3. *φ is an undischarged assumption of δ.*
4. *An inference with conclusion φ, upper derivations δ_1 (and δ_2, δ_3), and discharge label n is correct.*
5. *δ is a correct natural deduction derivation.*

Proof. We have to show that the corresponding relations between Gödel numbers of formulas, sequences of Gödel numbers of formulas (which code sets of formulas), and Gödel numbers of derivations are primitive recursive.

1. We want to show that $\mathrm{Assum}(x, d, n)$, which holds if x is the Gödel number of an assumption of the derivation with Gödel number d labelled n, is primitive recursive. For this we need a helper relation $\mathrm{hAssum}(x, d, n, i)$ which holds if the formula φ with Gödel number x occurs as an initial formula with label n in the derivation with Gödel number d within i inferences up from the end-formula.

$$\begin{aligned}
\mathrm{hAssum}(x, d, n, 0) &\Leftrightarrow \mathbb{T}\\
\mathrm{hAssum}(x, d, n, i+1) &\Leftrightarrow\\
&\qquad \mathrm{Sent}(x) \wedge (d = \langle 0, x, n\rangle \vee\\
&\qquad ((d)_0 = 1 \wedge \mathrm{hAssum}(x, (d)_4, n, i)) \vee\\
&\qquad ((d)_0 = 2 \wedge (\mathrm{hAssum}(x, (d)_4, n, i) \vee\\
&\qquad\quad \mathrm{hAssum}(x, (d)_5, n, i))) \vee\\
&\qquad ((d)_0 = 3 \wedge (\mathrm{hAssum}(x, (d)_3, n, i) \vee\\
&\qquad\quad \mathrm{hAssum}(x, (d)_2, n, i)) \vee \mathrm{hAssum}(x, (d)_3, n, i))
\end{aligned}$$

If the number i is large enough, e.g., larger than the maximum number of inferences between an initial formula and the end-formula of δ, it holds of x, d, n, and i iff φ is an initial formula in δ labelled n. The number d itself is larger than that maximum number of inferences. So we can define

$$\mathrm{Assum}(x, d, n) = \mathrm{hAssum}(x, d, n, d).$$

2. We want to show that $\mathrm{Discharge}(x, d, n)$, which holds if all assumptions with label n in the derivation with Gödel number d all are the formula with Gödel number x. But this relation holds iff $(\forall y < d)\ (\mathrm{Assum}(y, d, n) \rightarrow y = x)$.

3. An occurrence of an assumption is not open if it occurs with label n in a subderivation that ends in a rule with discharge label n. Define the helper relation $\mathrm{hNotOpen}(x, d, n, i)$ as

$$\begin{aligned}
\mathrm{hNotOpen}(x, d, n, 0) &\Leftrightarrow \mathbb{T} \\
\mathrm{hNotOpen}(x, d, n, i + 1) &\Leftrightarrow \\
&\quad (d)_2 = n \lor \\
&\quad ((d)_0 = 1 \land \mathrm{hNotOpen}(x, (d)_4, n, i)) \lor \\
&\quad ((d)_0 = 2 \land \mathrm{hNotOpen}(x, (d)_4, n, i) \land \\
&\qquad \mathrm{hNotOpen}(x, (d)_5, n, i))) \lor \\
&\quad ((d)_0 = 3 \land \mathrm{hNotOpen}(x, (d)_3, n, i) \land \\
&\qquad \mathrm{hNotOpen}(x, (d)_4, n, i) \land \mathrm{hNotOpen}(x, (d)_5, n, i))
\end{aligned}$$

Note that all assumptions of the form φ labelled n are discharged in δ iff either the last inference of δ discharges them (i.e., the last inference has label n), or if it is discharged in all of the immediate subderivations.

A formula φ is an open assumption of δ iff it is an initial formula of δ (with label n) and is not discharged in δ (by a rule with label n). We can then define $\mathrm{OpenAssum}(x, d)$ as

$$(\exists n < d)\ (\mathrm{Assum}(x, d, n, d) \land \lnot \mathrm{hNotOpen}(x, d, n, d)).$$

4. Here we have to show that for each rule of inference R the relation $\mathrm{FollowsBy}_R(x, d_1, n)$ which holds if x is the Gödel number of the conclusion and d_1 is the Gödel number of a derivation ending in the premise of a correct application of R with label n is primitive recursive, and similarly for rules with two or three premises.

The simplest case is that of the =Intro rule. Here there is no premise, i.e., $d_1 = 0$. However, φ must be of the form $t = t$, for a closed term t. Here, a primitive recursive definition is

$$(\exists t < x)\ (\mathrm{ClTerm}(t) \land x = ({}^{\#}{=}({}^{\#} \frown t \frown {}^{\#},{}^{\#} \frown t \frown {}^{\#}){}^{\#})) \land d_1 = 0).$$

For a more complicated example, $\mathrm{FollowsBy}_{\to\mathrm{Intro}}(x, d_1, n)$ holds iff φ is of the form $(\psi \to \chi)$, the end-formula of δ is χ, and any initial formula in δ labelled n is of the form ψ. We can express this primitive recursively by

$$\begin{aligned}
&(\exists y < x)\ (\mathrm{Sent}(y) \land \mathrm{Discharge}(y, d_1) \land \\
&\qquad\qquad (\exists z < x)\ (\mathrm{Sent}(y) \land (d)_1 = z) \land \\
&\qquad\qquad\qquad\qquad x = ({}^{\#}({}^{\#} \frown y \frown {}^{\#}{\to}^{\#} \frown z \frown {}^{\#}){}^{\#}))
\end{aligned}$$

(Think of y as the Gödel number of ψ and z as that of χ.)

For another example, consider $\exists$Intro. Here, φ is the conclusion of a correct inference with one upper derivation iff there is a formula ψ, a closed term t and a variable x such that $\psi[t/x]$ is the end-formula of the upper derivation and $\exists x\, \psi$ is the conclusion φ, i.e., the formula with Gödel number x. So $\mathrm{FollowsBy}_{\exists\mathrm{Intro}}(x, d_1, n)$ holds iff

$$\begin{aligned}
&\mathrm{Sent}(x) \land (\exists y < x)\ (\exists v < x)\ (\exists t < d)\ (\mathrm{Frm}(y) \land \mathrm{Term}(t) \land \mathrm{Var}(v) \land \\
&\qquad \mathrm{FreeFor}(y, t, v) \land \mathrm{Subst}(y, t, v) = (d_1)_1 \land x = ({}^{\#}\exists^{\#} \frown v \frown z))
\end{aligned}$$

5. We first define a helper relation $\mathrm{hDeriv}(d, i)$ which holds if d codes a correct derivation at least to i inferences up from the end sequent. $\mathrm{hDeriv}(d, 0)$ holds always. Otherwise, $\mathrm{hDeriv}(d, i+1)$ iff either d just codes an assumption or d ends in a correct inference and the codes of the immediate sub-derivations satisfy $\mathrm{hDeriv}(d', i)$.

$$\begin{aligned}
&\mathrm{hDeriv}(d, 0) \Leftrightarrow \mathbb{T} \\
&\mathrm{hDeriv}(d, i+1) \Leftrightarrow \\
&\qquad (\exists x < d)\ (\exists n < d)\ (\mathrm{Sent}(x) \land d = \langle 0, x, n\rangle) \lor \\
&\qquad ((d)_0 = 1 \land \\
&\qquad\quad ((d)_3 = 1 \land \mathrm{FollowsBy}_{\land\mathrm{Elim}}((d)_1, (d)_4, (d)_2) \lor \\
&\qquad\qquad \vdots \\
&\qquad\quad ((d)_3 = 10 \land \mathrm{FollowsBy}_{=\mathrm{Intro}}((d)_1, (d)_4, (d)_2)) \land \\
&\qquad\quad \mathrm{nDeriv}((d)_4, i)) \lor \\
&\qquad ((d)_0 = 2 \land \\
&\qquad\quad ((d)_3 = 1 \land \mathrm{FollowsBy}_{\land\mathrm{Intro}}((d)_1, (d)_4, (d)_5, (d)_2)) \lor \\
&\qquad\qquad \vdots \\
&\qquad\quad ((d)_3 = 3 \land \mathrm{FollowsBy}_{\lnot\mathrm{Elim}}((d)_1, (d)_4, (d)_5, (d)_2)) \land \\
&\qquad\quad \mathrm{hDeriv}((d)_4, i) \land \mathrm{hDeriv}((d)_5, i)) \lor \\
&\qquad ((d)_0 = 3 \land \\
&\qquad\quad \mathrm{FollowsBy}_{\lor\mathrm{Elim}}((d)_1, (d)_3, (d)_4, (d)_5, (d)_2) \land \\
&\qquad\quad \mathrm{hDeriv}((d)_3, i) \land \mathrm{hDeriv}((d)_4, i)) \land \mathrm{hDeriv}((d)_5, i)
\end{aligned}$$

This is a primitive recursive definition. Again we can define $\mathrm{Deriv}(d)$ as $\mathrm{hDeriv}(d, d)$.

□

Problem art.1. Define the following relations as in Proposition art.3:

1. $\mathrm{FollowsBy}_{\to\mathrm{Elim}}(x, d_1, d_2, n)$,
2. $\mathrm{FollowsBy}_{=\mathrm{Elim}}(x, d_1, d_2, n)$,
3. $\mathrm{FollowsBy}_{\lor\mathrm{Elim}}(x, d_1, d_2, d_3, n)$,
4. $\mathrm{FollowsBy}_{\forall\mathrm{Intro}}(x, d_1, n)$.

For the last one, you will have to also show that you can test primitive recursively if the formula with Gödel number x and the derivation with Gödel number d satisfy the eigenvariable condition, i.e., the eigenvariable a of the $\forall$Intro inference occurs neither in x nor in an open assumption of d.

Proposition art.4. *Suppose Γ is a primitive recursive set of sentences. Then the relation $\mathrm{Prf}_\Gamma(x, y)$ expressing "x is the code of a derivation δ of φ from undischarged assumptions in Γ and y is the Gödel number of φ" is primitive recursive.*

Proof. Suppose "$y \in \Gamma$" is given by the primitive recursive predicate $R_\Gamma(y)$. We have to show that $\mathrm{Prf}_\Gamma(x, y)$ which holds iff y is the Gödel number of a sentence φ and x is the code of a natural deduction derivation with end formula φ and all undischarged assumptions in Γ is primitive recursive.

By the previous proposition, the property $\mathrm{Deriv}(x)$ which holds iff x is the code of a correct derivation δ in natural deduction is primitive recursive. If x is such a code, then $(x)_1$ is the code of the end-formula of δ. Thus we can define $\mathrm{Prf}_\Gamma(x, y)$ by

$$\begin{aligned} \mathrm{Prf}_\Gamma(x, y) \Leftrightarrow{} & \mathrm{Deriv}(x) \land (x)_1 = y \land \\ & (\forall z < x)\,(\mathrm{OpenAssum}(z, x) \to R_\Gamma(z)) \end{aligned}$$

□

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Bibliography