

art.1 Derivations in Natural Deduction

inc:art:pnd:
sec In order to arithmetize **derivations**, we must represent **derivations** as numbers. explanation Since **derivations** are trees of **formulas** where each inference carries one or two labels, a recursive representation is the most obvious approach: we represent a **derivation** as a tuple, the components of which are the number of immediate sub-**derivations** leading to the premises of the last inference, the representations of these sub-**derivations**, and the end-**formula**, the discharge label of the last inference, and a number indicating the type of the last inference.

Definition art.1. If δ is a **derivation** in natural deduction, then $\# \delta^\#$ is defined inductively as follows:

1. If δ consists only of the assumption φ , then $\# \delta^\#$ is $\langle 0, \# \varphi^\#, n \rangle$. The number n is 0 if it is an **undischarged** assumption, and the numerical label otherwise.
2. If δ ends in an inference with one, two, or three premises, then $\# \delta^\#$ is

$$\begin{aligned} &\langle 1, \# \delta_1^\#, \# \varphi^\#, n, k \rangle, \\ &\langle 2, \# \delta_1^\#, \# \delta_2^\#, \# \varphi^\#, n, k \rangle, \text{ or} \\ &\langle 3, \# \delta_1^\#, \# \delta_2^\#, \# \delta_3^\#, \# \varphi^\#, n, k \rangle, \end{aligned}$$

respectively. Here $\delta_1, \delta_2, \delta_3$ are the sub-**derivations** ending in the premise(s) of the last inference in δ , φ is the conclusion of the last inference in δ , n is the discharge label of the last inference (0 if the inference does not discharge any assumptions), and k is given by the following table according to which rule was used in the last inference.

Rule:	\wedge Intro	\wedge Elim	\vee Intro	\vee Elim
k :	1	2	3	4

Rule:	\rightarrow Intro	\rightarrow Elim	\neg Intro	\neg Elim
k :	5	6	7	8

Rule:	\perp_I	\perp_C	\forall Intro	\forall Elim
k :	9	10	11	12

Rule:	\exists Intro	\exists Elim	$=$ Intro	$=$ Elim
k :	13	14	15	16

Example art.2. Consider the very simple **derivation**

$$1 \frac{\frac{[\varphi \wedge \psi]^1}{\varphi} \wedge\text{Elim}}{(\varphi \wedge \psi) \rightarrow \varphi} \rightarrow\text{Intro}$$

The Gödel number of the assumption would be $d_0 = \langle 0, \# \varphi \wedge \psi^\#, 1 \rangle$. The Gödel number of the **derivation** ending in the conclusion of \wedge Elim would

be $d_1 = \langle 1, d_0, \# \varphi^\#, 0, 2 \rangle$ (1 since \wedge Elim has one premise, the Gödel number of conclusion φ , 0 because no assumption is discharged, and 2 is the number coding \wedge Elim). The Gödel number of the entire **derivation** then is $\langle 1, d_1, \#((\varphi \wedge \psi) \rightarrow \varphi)^\#, 1, 5 \rangle$, i.e.,

$$\langle 1, \langle 1, \langle 0, \#(\varphi \wedge \psi)^\#, 1 \rangle, \# \varphi^\#, 0, 2 \rangle, \#((\varphi \wedge \psi) \rightarrow \varphi)^\#, 1, 5 \rangle.$$

explanation

Having settled on a representation of **derivations**, we must also show that we can manipulate Gödel numbers of such **derivations** primitive recursively, and express their essential properties and relations. Some operations are simple: e.g., given a Gödel number d of a **derivation**, $\text{EndFmla}(d) = (d)_{(d)_0+1}$ gives us the Gödel number of its end-**formula**, $\text{DischargeLabel}(d) = (d)_{(d)_0+2}$ gives us the discharge label and $\text{LastRule}(d) = (d)_{(d)_0+3}$ the number indicating the type of the last inference. Some are much harder. We'll at least sketch how to do this. The goal is to show that the relation “ δ is a **derivation** of φ from Γ ” is a primitive recursive relation of the Gödel numbers of δ and φ .

Proposition art.3. *The following relations are primitive recursive:*

1. φ occurs as an assumption in δ with label n .
2. All assumptions in δ with label n are of the form φ (i.e., we can **discharge** the assumption φ using label n in δ).

Proof. We have to show that the corresponding relations between Gödel numbers of **formulas** and Gödel numbers of **derivations** are primitive recursive.

1. We want to show that $\text{Assum}(x, d, n)$, which holds if x is the Gödel number of an assumption of the **derivation** with Gödel number d labelled n , is primitive recursive. This is the case if the **derivation** with Gödel number $\langle 0, x, n \rangle$ is a sub-**derivation** of d . Note that the way we code **derivations** is a special case of the coding of trees introduced in ??, so the primitive recursive function $\text{SubtreeSeq}(d)$ gives a sequence of Gödel numbers of all sub-**derivations** of d (of length at most d). So we can define

$$\text{Assum}(x, d, n) \Leftrightarrow (\exists i < d) (\text{SubtreeSeq}(d))_i = \langle 0, x, n \rangle.$$

2. We want to show that $\text{Discharge}(x, d, n)$, which holds if all assumptions with label n in the **derivation** with Gödel number d all are the **formula** with Gödel number x . But this relation holds iff $(\forall y < d) (\text{Assum}(y, d, n) \rightarrow y = x)$. \square

Proposition art.4. *The property $\text{Correct}(d)$ which holds iff the last inference in the **derivation** δ with Gödel number d is correct, is primitive recursive.*

*inc:art:pnd:
prop:followsby*

Proof. Here we have to show that for each rule of inference R the relation $\text{FollowsBy}_R(d)$ is primitive recursive, where $\text{FollowsBy}_R(d)$ holds iff d is the Gödel number of **derivation** δ , and the end-**formula** of δ follows by a correct application of R from the immediate sub-**derivations** of δ .

A simple case is that of the $\wedge\text{Intro}$ rule. If δ ends in a correct $\wedge\text{Intro}$ inference, it looks like this:

$$\frac{\begin{array}{c} \vdots \\ \delta_1 \\ \vdots \\ \varphi \end{array} \quad \begin{array}{c} \vdots \\ \delta_2 \\ \vdots \\ \psi \end{array}}{\varphi \wedge \psi} \wedge\text{Intro}$$

Then the Gödel number d of δ is $\langle 2, d_1, d_2, \#(\varphi \wedge \psi)\#, 0, k \rangle$ where $\text{EndFmla}(d_1) = \# \varphi \#, \text{EndFmla}(d_2) = \# \psi \#, n = 0$, and $k = 1$. So we can define $\text{FollowsBy}_{\wedge\text{Intro}}(d)$ as

$$(d)_0 = 2 \wedge \text{DischargeLabel}(d) = 0 \wedge \text{LastRule}(d) = 1 \wedge \\ \text{EndFmla}(d) = \#(\# \frown \text{EndFmla}((d)_1) \frown \# \wedge \# \frown \text{EndFmla}((d)_2) \frown \#)\#.$$

Another simple example is the $=\text{Intro}$ rule. Here the premise is an empty **derivation**, i.e., $(d)_1 = 0$, and no discharge label, i.e., $n = 0$. However, φ must be of the form $t = t$, for a closed term t . Here, a primitive recursive definition is

$$(d)_0 = 1 \wedge (d)_1 = 0 \wedge \text{DischargeLabel}(d) = 0 \wedge \\ (\exists t < d) (\text{CIterm}(t) \wedge \text{EndFmla}(d) = \#(=\# \frown t \frown \#, \# \frown t \frown \#)\#)$$

For a more complicated example, $\text{FollowsBy}_{\rightarrow\text{Intro}}(d)$ holds iff the end-**formula** of δ is of the form $(\varphi \rightarrow \psi)$, where the end-**formula** of δ_1 is ψ , and any assumption in δ labelled n is of the form φ . We can express this primitive recursively by

$$(d)_0 = 1 \wedge \\ (\exists a < d) (\text{Discharge}(a, (d)_1, \text{DischargeLabel}(d)) \wedge \\ \text{EndFmla}(d) = (\#(\# \frown a \frown \# \rightarrow \# \frown \text{EndFmla}((d)_1) \frown \#)\#))$$

(Think of a as the Gödel number of φ).

For another example, consider $\exists\text{Intro}$. Here, the last inference in δ is correct iff there is a **formula** φ , a closed term t and a **variable** x such that $\varphi[t/x]$ is the end-**formula** of the **derivation** δ_1 and $\exists x \varphi$ is the conclusion of the last inference. So, $\text{FollowsBy}_{\exists\text{Intro}}(d)$ holds iff

$$(d)_0 = 1 \wedge \text{DischargeLabel}(d) = 0 \wedge \\ (\exists a < d) (\exists x < d) (\exists t < d) (\text{CIterm}(t) \wedge \text{Var}(x) \wedge \\ \text{Subst}(a, t, x) = \text{EndFmla}((d)_1) \wedge \text{EndFmla}(d) = (\# \exists \# \frown x \frown a)).$$

We then define $\text{Correct}(d)$ as

$$\begin{aligned} \text{Sent}(\text{EndFmla}(d)) \wedge \\ (\text{LastRule}(d) = 1 \wedge \text{FollowsBy}_{\wedge\text{Intro}}(d)) \vee \cdots \vee \\ (\text{LastRule}(d) = 16 \wedge \text{FollowsBy}_{=\text{Elim}}(d)) \vee \\ (\exists n < d) (\exists x < d) (d = \langle 0, x, n \rangle). \end{aligned}$$

The first line ensures that the end-formula of d is a sentence. The last line covers the case where d is just an assumption. \square

Problem art.1. Define the following properties as in [Proposition art.4](#):

1. $\text{FollowsBy}_{\rightarrow\text{Elim}}(d)$,
2. $\text{FollowsBy}_{=\text{Elim}}(d)$,
3. $\text{FollowsBy}_{\vee\text{Elim}}(d)$,
4. $\text{FollowsBy}_{\vee\text{Intro}}(d)$.

For the last one, you will have to also show that you can test primitive recursively if the last inference of the derivation with Gödel number d satisfies the eigenvariable condition, i.e., the eigenvariable a of the $\forall\text{Intro}$ inference occurs neither in the end-formula of d nor in an open assumption of d . You may use the primitive recursive predicate OpenAssum from [Proposition art.6](#) for this.

Proposition art.5. *The relation $\text{Deriv}(d)$ which holds if d is the Gödel number of a correct derivation δ , is primitive recursive.* inc:art:pnd:
prop:deriv

Proof. A derivation δ is correct if every one of its inferences is a correct application of a rule, i.e., if every one of its sub-derivations ends in a correct inference. So, $\text{Deriv}(d)$ iff

$$(\forall i < \text{len}(\text{SubtreeSeq}(d))) \text{Correct}((\text{SubtreeSeq}(d))_i) \quad \square$$

Proposition art.6. *The relation $\text{OpenAssum}(z, d)$ that holds if z is the Gödel number of an undischarged assumption φ of the derivation δ with Gödel number d , is primitive recursive.* inc:art:pnd:
prop:openassum

Proof. An occurrence of an assumption is discharged if it occurs with label n in a sub-derivation of δ that ends in a rule with discharge label n . So φ is an undischarged assumption of δ if at least one of its occurrences is not discharged in δ . We must be careful: δ may contain both discharged and undischarged occurrences of φ .

Consider a sequence $\delta_0, \dots, \delta_k$ where $\delta_0 = \delta$, δ_k is the assumption $[\varphi]^n$ (for some n), and δ_{i+1} is an immediate sub-derivation of δ_i . If such a sequence exists in which no δ_i ends in an inference with discharge label n , then φ is an undischarged assumption of δ .

The primitive recursive function $\text{SubtreeSeq}(d)$ provides us with a sequence of Gödel numbers of all sub-derivations of δ . Any sequence of Gödel numbers of sub-derivations of δ is a subsequence of it. Being a subsequence of is a primitive recursive relation: $\text{Subseq}(s, s')$ holds iff $(\forall i < \text{len}(s)) \exists j < \text{len}(s') (s)_i = (s')_j$. Being an immediate sub-derivation is as well: $\text{Subderiv}(d, d')$ iff $(\exists j < (d')_0) d = (d')_j$. So we can define $\text{OpenAssum}(z, d)$ by

$$\begin{aligned} (\exists s < \text{SubtreeSeq}(d)) (\text{Subseq}(s, \text{SubtreeSeq}(d)) \wedge (s)_0 = d \wedge \\ (\exists n < d) ((s)_{\text{len}(s)-1} = \langle 0, z, n \rangle \wedge \\ (\forall i < (\text{len}(s) - 1)) (\text{Subderiv}((s)_{i+1}, (s)_i)] \wedge \\ \text{DischargeLabel}((s)_{i+1} \neq n))). \quad \square \end{aligned}$$

Proposition art.7. *Suppose Γ is a primitive recursive set of sentences. Then the relation $\text{Prf}_\Gamma(x, y)$ expressing “ x is the code of a derivation δ of φ from undischarged assumptions in Γ and y is the Gödel number of φ ” is primitive recursive.*

Proof. Suppose “ $y \in \Gamma$ ” is given by the primitive recursive predicate $R_\Gamma(y)$. We have to show that $\text{Prf}_\Gamma(x, y)$ which holds iff y is the Gödel number of a sentence φ and x is the code of a natural deduction derivation with end formula φ and all undischarged assumptions in Γ is primitive recursive.

By **Proposition art.5**, the property $\text{Deriv}(x)$ which holds iff x is the Gödel number of a correct derivation δ in natural deduction is primitive recursive. Thus we can define $\text{Prf}_\Gamma(x, y)$ by

$$\begin{aligned} \text{Prf}_\Gamma(x, y) \Leftrightarrow \text{Deriv}(x) \wedge \text{EndFmla}(x) = y \wedge \\ (\forall z < x) (\text{OpenAssum}(z, x) \rightarrow R_\Gamma(z)). \quad \square \end{aligned}$$

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Bibliography