

art.1 Derivations in Natural Deduction

inc:art:pnd:
sec In order to arithmetize **derivations**, we must represent **derivations** as numbers. explanation
 Since **derivations** are trees of **formulas** where each inference carries one or two labels, a recursive representation is the most obvious approach: we represent a **derivation** as a tuple, the components of which are the number of immediate sub-**derivations** leading to the premises of the last inference, the representations of these sub-**derivations**, and the end-**formula**, the discharge label of the last inference, and a number indicating the type of the last inference.

Definition art.1. If δ is a **derivation** in natural deduction, then $\# \delta^\#$ is defined inductively as follows:

1. If δ consists only of the assumption φ , then $\# \delta^\#$ is $\langle 0, \# \varphi^\#, n \rangle$. The number n is 0 if it is an **undischarged** assumption, and the numerical label otherwise.
2. If δ ends in an inference with one, two, or three premises, then $\# \delta^\#$ is

$$\begin{aligned} &\langle 1, \# \delta_1^\#, \# \varphi^\#, n, k \rangle, \\ &\langle 2, \# \delta_1^\#, \# \delta_2^\#, \# \varphi^\#, n, k \rangle, \text{ or} \\ &\langle 3, \# \delta_1^\#, \# \delta_2^\#, \# \delta_3^\#, \# \varphi^\#, n, k \rangle, \end{aligned}$$

respectively. Here $\delta_1, \delta_2, \delta_3$ are the sub-**derivations** ending in the premise(s) of the last inference in δ , φ is the conclusion of the last inference in δ , n is the discharge label of the last inference (0 if the inference does not discharge any assumptions), and k is given by the following table according to which rule was used in the last inference.

Rule:	\wedge Intro	\wedge Elim	\vee Intro	\vee Elim
k :	1	2	3	4
Rule:	\rightarrow Intro	\rightarrow Elim	\neg Intro	\neg Elim
k :	5	6	7	8
Rule:	\perp_I	\perp_C	\forall Intro	\forall Elim
k :	9	10	11	12
Rule:	\exists Intro	\exists Elim	$=$ Intro	$=$ Elim
k :	13	14	15	16

Example art.2. Consider the very simple derivation

$${}^1 \frac{\frac{[\varphi \wedge \psi]^1}{\varphi} \wedge \text{Elim}}{(\varphi \wedge \psi) \rightarrow \varphi} \rightarrow \text{Intro}$$

The Gödel number of the assumption would be $d_0 = \langle 0, \# \varphi \wedge \psi^\#, 1 \rangle$. The Gödel number of the derivation ending in the conclusion of \wedge Elim would

be $d_1 = \langle 1, d_0, \# \varphi^\#, 0, 2 \rangle$ (1 since \wedge Elim has one premise, the Gödel number of conclusion φ , 0 because no assumption is discharged, and 2 is the number coding \wedge Elim). The Gödel number of the entire derivation then is $\langle 1, d_1, \#((\varphi \wedge \psi) \rightarrow \varphi)^\#, 1, 5 \rangle$, i.e.,

$$\langle 1, \langle 1, \langle 0, \#(\varphi \wedge \psi)^\#, 1 \rangle, \# \varphi^\#, 0, 2 \rangle, \#((\varphi \wedge \psi) \rightarrow \varphi)^\#, 1, 5 \rangle.$$

explanation

Having settled on a representation of **derivations**, we must also show that we can manipulate Gödel numbers of such **derivations** primitive recursively, and express their essential properties and relations. Some operations are simple: e.g., given a Gödel number d of a **derivation**, $\text{EndFmla}(d) = (d)_{(d)_0+1}$ gives us the Gödel number of its end-**formula**, $\text{DischargeLabel}(d) = (d)_{(d)_0+2}$ gives us the discharge label and $\text{LastRule}(d) = (d)_{(d)_0+3}$ the number indicating the type of the last inference. Some are much harder. We'll at least sketch how to do this. The goal is to show that the relation “ δ is a **derivation** of φ from Γ ” is a primitive recursive relation of the Gödel numbers of δ and φ .

Proposition art.3. *The following relations are primitive recursive:*

*inc:art:pnd:
prop:followsby*

1. φ occurs as an assumption in δ with label n .
2. All assumptions in δ with label n are of the form φ (i.e., we can **discharge** the assumption φ using label n in δ).

Proof. We have to show that the corresponding relations between Gödel numbers of **formulas** and Gödel numbers of **derivations** are primitive recursive.

1. We want to show that $\text{Assum}(x, d, n)$, which holds if x is the Gödel number of an assumption of the **derivation** with Gödel number d labelled n , is primitive recursive. This is the case if the **derivation** with Gödel number $\langle 0, x, n \rangle$ is a sub-**derivation** of d . Note that the way we code derivations is a special case of the coding of trees introduced in ??, so the primitive recursive function $\text{SubtreeSeq}(d)$ gives a sequence of Gödel numbers of all sub-**derivations** of d (of length at most d). So we can define

$$\text{Assum}(x, d, n) \Leftrightarrow (\exists i < d) (\text{SubtreeSeq}(d))_i = \langle 0, x, n \rangle.$$

2. We want to show that $\text{Discharge}(x, d, n)$, which holds if all assumptions with label n in the **derivation** with Gödel number d all are the **formula** with Gödel number x . But this relation holds iff $(\forall y < d) (\text{Assum}(y, d, n) \rightarrow y = x)$.

□

Proposition art.4. *The relation $\text{Correct}(d)$ which holds if the last inference in the derivation δ with Gödel number d is correct, is primitive recursive.*

Proof. Here we have to show that for each rule of inference R the relation $\text{FollowsBy}_R(d)$ is primitive recursive, where $\text{FollowsBy}_R(d)$ holds iff d is the Gödel number of **derivation** δ , and the end-**formula** of δ follows by a correct application of R from the immediate sub-**derivations** of δ .

A simple case is that of the $\wedge\text{Intro}$ rule. If δ ends in a correct $\wedge\text{Intro}$ inference, it looks like this:

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ A \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ B \end{array}}{A \wedge B} \wedge\text{Intro}$$

Then the Gödel number d of δ is $\langle 2, d_1, d_2, \#(A \wedge B)^\#, 0, k \rangle$ where $\text{EndFmla}(d_1) = \#A^\#, \text{EndFmla}(d_2) = \#B^\#, n = 0$, and $k = 1$. So we can define $\text{FollowsBy}_{\wedge\text{Intro}}(d)$ as

$$(d)_0 = 2 \wedge \text{DischargeLabel}(d) = 0 \wedge \text{LastRule}(d) = 1 \wedge \\ \text{EndFmla}(d) = \#(\# \frown \text{EndFmla}((d)_1) \frown \# \wedge \# \frown \text{EndFmla}((d)_2) \frown \#)^\#.$$

Another simple example is the $=\text{Intro}$ rule. Here the premise is an empty derivation, i.e., $(d)_1 = 0$, and no discharge label, i.e., $n = 0$. However, φ must be of the form $t = t$, for a closed term t . Here, a primitive recursive definition is

$$(d)_0 = 1 \wedge (d)_1 = 0 \wedge \text{DischargeLabel}(d) = 0 \wedge \\ (\exists t < d) (\text{ClTerm}(t) \wedge \text{EndFmla}(d) = \#(=\# \frown t \frown \# \frown t \frown \#)^\#)$$

For a more complicated example, $\text{FollowsBy}_{\rightarrow\text{Intro}}(d)$ holds iff the end-**formula** of δ is of the form $(\varphi \rightarrow \psi)$, where the end-**formula** of δ_1 is ψ , and any assumption in δ labelled n is of the form φ . We can express this primitive recursively by

$$(d)_0 = 1 \wedge \\ (\exists a < d) (\text{Discharge}(a, (d)_1, \text{DischargeLabel}(d)) \wedge \\ \text{EndFmla}(d) = (\#(\# \frown a \frown \# \rightarrow \# \frown \text{EndFmla}((d)_1) \frown \#)^\#))$$

(Think of a as the Gödel number of φ).

For another example, consider $\exists\text{Intro}$. Here, the last inference in δ is correct iff there is a **formula** φ , a closed term t and a **variable** x such that $\varphi[t/x]$ is the end-**formula** of the derivation δ_1 and $\exists x \varphi$ is the conclusion of the last inference. So, $\text{FollowsBy}_{\exists\text{Intro}}(d)$ holds iff

$$(d)_0 = 1 \wedge \text{DischargeLabel}(d) = 0 \wedge \\ (\exists a < d) (\exists x < d) (\exists t < d) (\text{ClTerm}(t) \wedge \text{Var}(x) \wedge \\ \text{Subst}(a, t, x) = \text{EndFmla}((d)_1) \wedge \text{EndFmla}(d) = (\#\exists\# \frown x \frown a))$$

We then define $\text{Correct}(d)$ as

$$\begin{aligned} \text{Sent}(\text{EndFmla}(d)) \wedge (\text{LastRule}(d) = 1 \wedge \text{FollowsBy}_{\wedge\text{Intro}}(d)) \vee \dots \vee \\ (\text{LastRule}(d) = 16 \wedge \text{FollowsBy}_{=\text{Elim}}(d)) \vee \\ (\exists n < d) (\exists x < d) (d = \langle 0, x, n \rangle) \end{aligned}$$

The first line ensures that the end-**formula** of d is a sentence. The last line covers the case where d is just an assumption. \square

Problem art.1. Define the following relations as in [Proposition art.3](#):

1. $\text{FollowsBy}_{\rightarrow\text{Elim}}(d)$,
2. $\text{FollowsBy}_{=\text{Elim}}(d)$,
3. $\text{FollowsBy}_{\vee\text{Elim}}(d)$,
4. $\text{FollowsBy}_{\forall\text{Intro}}(d)$.

For the last one, you will have to also show that you can test primitive recursively if the the last inference of the **derivation** with Gödel number d satisfies the eigenvariable condition, i.e., the eigenvariable a of the $\forall\text{Intro}$ inference occurs neither in the end-**formula** of d nor in an open assumption of d . You may use the primitive recursive predicate OpenAssum from [Proposition art.6](#) for this.

Proposition art.5. *The relation $\text{Deriv}(d)$ which holds if d is the Gödel number of a correct **derivation** δ , is primitive recursive.*

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prop:deriv*

Proof. A **derivation** δ is correct if every one of its inferences is a correct application of a rule, i.e., if every one of its sub-**derivations** ends in a correct inference. So, $\text{Deriv}(d)$ iff

$$(\forall i < \text{len}(\text{SubtreeSeq}(d))) \text{Correct}((\text{SubtreeSeq}(d))_i)$$

\square

Proposition art.6. *The relation $\text{OpenAssum}(z, d)$ that holds if z is the Gödel number of an **undischarged** assumption φ of the derivation δ with Gödel number d , is primitive recursive.*

*inc:art:pnd:
prop:openassum*

Proof. An occurrence of an assumption is **discharged** if it occurs with label n in a sub-**derivation** of δ that ends in a rule with discharge label n . So φ is an **undischarged** assumption of δ if at least one of its occurrences is not **discharged** in δ . We must be careful: δ may contain both **discharged** and **undischarged** occurrences of φ .

Consider a sequence $\delta_0, \dots, \delta_k$ where $\delta_0 = d$, δ_k is the assumption $[\varphi]^n$ (for some n), and δ_i is an immediate sub-**derivation** of δ_{i+1} . If such a sequence

exists in which no δ_i ends in an inference with discharge label n , then φ is an **undischarged** assumption of δ .

The primitive recursive function $\text{SubtreeSeq}(d)$ provides us with a sequence of Gödel numbers of all sub-**derivations** of δ . Any sequence of Gödel numbers of sub-**derivations** of δ is a subsequence of it. Being a subsequence of is a primitive recursive relation: $\text{Subseq}(s, s')$ holds iff $(\forall i < \text{len}(s)) \exists j < \text{len}(s') (s)_i = (s')_j$. Being an immediate sub-**derivation** is as well: $\text{Subderiv}(d, d')$ iff $(\exists j < (d')_0) d = (d')_j$. So we can define $\text{OpenAssum}(z, d)$ by

$$\begin{aligned} (\exists s < \text{SubtreeSeq}(d)) & (\text{Subseq}(s, \text{SubtreeSeq}(d)) \wedge (s)_0 = d \wedge \\ & (\exists n < d) ((s)_{\text{len}(s)-1} = \langle 0, z, n \rangle \wedge \\ & (\forall i < (\text{len}(s) - 1)) (\text{Subderiv}((s)_i, (s)_{i+1}) \wedge \\ & \text{DischargeLabel}((s)_{i+1}) \neq n)). \end{aligned}$$

□

Proposition art.7. *Suppose Γ is a primitive recursive set of **sentences**. Then the relation $\text{Prf}_\Gamma(x, y)$ expressing “ x is the code of a **derivation** δ of φ from **undischarged** assumptions in Γ and y is the Gödel number of φ ” is primitive recursive.*

Proof. Suppose “ $y \in \Gamma$ ” is given by the primitive recursive predicate $R_\Gamma(y)$. We have to show that $\text{Prf}_\Gamma(x, y)$ which holds iff y is the Gödel number of a sentence φ and x is the code of a natural deduction **derivation** with end **formula** φ and all **undischarged** assumptions in Γ is primitive recursive.

By [Proposition art.5](#), the property $\text{Deriv}(x)$ which holds iff x is the Gödel number of a correct derivation δ in natural deduction is primitive recursive. Thus we can define $\text{Prf}_\Gamma(x, y)$ by

$$\begin{aligned} \text{Deriv}(x) \wedge \text{EndFmla}(x) = y \wedge \\ (\forall z < x) (\text{OpenAssum}(z, x) \rightarrow R_\Gamma(z)) \end{aligned}$$

□

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Bibliography