In order to arithmetize derivations, we must represent derivations as numbers. Since derivations are trees of sequents where each inference carries also a label, a recursive representation is the most obvious approach: we represent a derivation as a tuple, the components of which are the end-sequent, the label, and the representations of the sub-derivations leading to the premises of the last inference.

**Definition art.1.** If \( \Gamma \) is a finite sequence of sentences, \( \Gamma = \langle \varphi_1, \ldots, \varphi_n \rangle \), then \( ^*\Gamma^* = \langle ^*\varphi_1^*, \ldots, ^*\varphi_n^* \rangle \).

If \( \Gamma \Rightarrow \Delta \) is a sequent, then a Gödel number of \( \Gamma \Rightarrow \Delta \) is

\[
^*\Gamma \Rightarrow \Delta^* = \langle ^*\Gamma^*, ^*\Delta^* \rangle
\]

If \( \pi \) is a derivation in LK, then \( ^*\pi^* \) is defined as follows:

1. If \( \pi \) consists only of the initial sequent \( \Gamma \Rightarrow \Delta \), then \( ^*\pi^* \) is

\[
\langle 0, ^*\Gamma \Rightarrow \Delta^* \rangle.
\]

2. If \( \pi \) ends in an inference with one or two premises, has \( \Gamma \Rightarrow \Delta \) as its conclusion, and \( \pi_1 \) and \( \pi_2 \) are the immediate subproof ending in the premise of the last inference, then \( ^*\pi^* \) is

\[
\langle 1, ^*\pi_1^*, ^*\Gamma \Rightarrow \Delta^*, k \rangle \text{ or } \langle 2, ^*\pi_1^*, ^*\pi_2^*, ^*\Gamma \Rightarrow \Delta^*, k \rangle,
\]

respectively, where \( k \) is given by the following table according to which rule was used in the last inference:

<table>
<thead>
<tr>
<th>Rule</th>
<th>WL</th>
<th>WR</th>
<th>CL</th>
<th>CR</th>
<th>XL</th>
<th>XR</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rule</th>
<th>( \neg )</th>
<th>( \neg )</th>
<th>( \wedge L )</th>
<th>( \wedge R )</th>
<th>( \vee L )</th>
<th>( \vee R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rule</th>
<th>( \rightarrow L )</th>
<th>( \rightarrow R )</th>
<th>( \forall L )</th>
<th>( \forall R )</th>
<th>( \exists L )</th>
<th>( \exists R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rule</th>
<th>Cut</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>19</td>
</tr>
</tbody>
</table>

**Example art.2.** Consider the very simple derivation

\[
\varphi \Rightarrow \varphi \quad \varphi \wedge \psi \Rightarrow \varphi \quad \wedge L
\]

\[
\Rightarrow (\varphi \wedge \psi) \Rightarrow \varphi \quad \rightarrow R
\]
Having settled on a representation of derivations, we must also show that we can manipulate such derivations primitive recursively, and express their essential properties and relations so. Some operations are simple: e.g., given a Gödel number $p$ of a derivation, $\text{EndSeq}(p) = (p)_{p+1}$ gives us the Gödel number of its end-sequent and $\text{LastRule}(p) = (p)_{p+2}$ the code of its last rule.

The property $\text{Sequent}(s)$ defined by

$$\text{len}(s) = 2 \land (\forall i < \text{len}(s)) \land \text{Sent}((s)_0 \land (s)_1)$$

holds of $s$ iff $s$ is the Gödel number of a sequent consisting of sentences. Some are much harder. We’ll at least sketch how to do this. The goal is to show that the relation "$p$ is a derivation of $\varphi$ from $\Gamma$" is a primitive recursive relation of the Gödel numbers of $\pi$ and $\varphi$.

**Proposition art.3.** The property $\text{Correct}(p)$ which holds iff the last inference in the derivation $\pi$ with Gödel number $p$ is correct, is primitive recursive.

**Proof.** $\Gamma \Rightarrow \Delta$ is an initial sequent if either there is a sentence $\varphi$ such that $\Gamma \Rightarrow \Delta$ is $\varphi \Rightarrow \varphi$, or there is a term $t$ such that $\Gamma \Rightarrow \Delta$ is $\emptyset \Rightarrow t = t$. In terms of Gödel numbers, $\text{InitSeq}(s)$ holds iff

$$\exists x < s \exists \langle x \rangle \text{Sent}(\langle x \rangle) \land s = \langle \langle x \rangle, \langle (x) \rangle \rangle$$

$$\exists t < s \exists \langle t \rangle \text{Term}(\langle t \rangle) \land s = \langle 0, \langle \# \text{ t} \rangle \rangle$$

We also have to show that for each rule of inference $R$ the relation $\text{FollowsBy}_R(p)$ is primitive recursive, where $\text{FollowsBy}_R(p)$ holds iff $p$ is the Gödel number of derivation $\pi$, and the end-sequent of $\pi$ follows by a correct application of $R$ from the immediate sub-derivations of $\pi$.

A simple case is that of the $\land R$ rule. If $\pi$ ends in a correct $\land R$ inference, it looks like this:

$$\vdots \vdots \vdots \vdots \vdots$$

$$\pi_1 \quad \pi_2$$

$$\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi \quad \land R$$

So, the last inference in the derivation $\pi$ is a correct application of $\land R$ iff there are sequences of sentences $\Gamma$ and $\Delta$ as well as two sentences $\varphi$ and $\psi$ such that the end-sequent of $\pi_1$ is $\Gamma \Rightarrow \Delta, \varphi$, the end-sequent of $\pi_2$ is $\Gamma \Rightarrow \Delta, \psi$, and...
and the end-sequent of $\pi$ is $\Gamma \Rightarrow \Delta, \varphi \land \psi$. We just have to translate this into Gödel numbers. If $s = \# \Gamma \Rightarrow \Delta \# \varphi \# \psi \#$ then $(s)_0 = \# \Gamma \#$ and $(s)_1 = \# \Delta \#$. So, $\text{FollowsBy}_{\land R}(p)$ holds iff

$$\exists g < p \ (\exists d < p \ (\exists a < p \ (\exists b < p) \text{ EndSequent}(p) = \langle g, d \vdash \langle \# \varphi \# \psi \rangle \rangle \land$$

$$\text{EndSequent}((p)_1) = \langle g, d \vdash \langle a \rangle \rangle \land$$

$$\text{EndSequent}((p)_2) = \langle g, d \vdash \langle b \rangle \rangle \land$$

$$(p)_0 = 2 \land \text{LastRule}(p) = 10.$$

The individual lines express, respectively, “there is a sequence ($\Gamma$) with Gödel number $g$, there is a sequence ($\Delta$) with Gödel number $d$, a formula ($\varphi$) with Gödel number $a$, and a formula ($\psi$) with Gödel number $b$,” such that “the end-sequent of $\pi$ is $\Gamma \Rightarrow \Delta, \varphi \land \psi$,” “the end-sequent of $\pi_1$ is $\Gamma \Rightarrow \Delta, \varphi$,” “the end-sequent of $\pi_2$ is $\Gamma \Rightarrow \Delta, \psi$,” and “$\pi$ has two immediate subderivations and the last inference rule is $\land R$ (with number 10).”

The last inference in $\pi$ is a correct application of $\exists R$ iff there are sequences $\Gamma$ and $\Delta$, a formula $\varphi$, a variable $x$, and a term $t$, such that the end-sequent of $\pi$ is $\Gamma \Rightarrow \Delta, \exists x \varphi$ and the end-sequent of $\pi_1$ is $\Gamma \Rightarrow \Delta, \varphi[t/x]$. So in terms of Gödel numbers, we have $\text{FollowsBy}_{\exists R}(p)$ iff

$$(\exists g < p) \ (\exists d < p) \ (\exists a < p) \ (\exists x < p) \ (\exists t < p) \text{ EndSequent}(p) = \langle g, d \vdash \langle \exists \varphi \rangle \rangle \land$$

$$\text{EndSequent}((p)_1) = \langle g, d \vdash \langle \text{Subst}(a, t, x) \rangle \rangle \land$$

$$(p)_0 = 1 \land \text{LastRule}(p) = 18.$$

We then define $\text{Correct}(p)$ as

$$\text{Sequent}(\text{EndSequent}(p)) \land$$

$$[(\text{LastRule}(p) = 1 \land \text{FollowsBy}_\land L(p)) \lor \cdots \lor$$

$$(\text{LastRule}(p) = \# \land \text{FollowsBy}_= (p)) \lor$$

$$(p)_0 = 0 \land \text{InitialSeq}([\text{EndSequent}(p)])]$$

The first line ensures that the end-sequent of $d$ is actually a sequent consisting of sentences. The last line covers the case where $p$ is just an initial sequent. □

**Problem art.1.** Define the following properties as in Proposition art.3:

1. $\text{FollowsBy}_{\land L}(p)$,
2. $\text{FollowsBy}_{\land R}(p)$,
3. $\text{FollowsBy}_=(p)$,
4. $\text{FollowsBy}_{\forall R}(p)$.
For the last one, you will have to also show that you can test primitive recursively if the last inference of the derivation with Gödel number \( p \) satisfies the eigenvariable condition, i.e., the eigenvariable \( a \) of the \( \forall R \) does not occur in the end-sequent.

**Proposition art.4.** The relation \( \text{Deriv}(p) \) which holds if \( p \) is the Gödel number of a correct derivation \( \pi \), is primitive recursive.

**Proof.** A derivation \( \pi \) is correct if every one of its inferences is a correct application of a rule, i.e., if every one of its sub-derivations ends in a correct inference. So, \( \text{Deriv}(d) \) iff

\[
\forall i < \text{len}((\text{SubtreeSeq}(p))) \ \text{Correct}((\text{SubtreeSeq}(p)))_i.
\]

**Proposition art.5.** Suppose \( \Gamma \) is a primitive recursive set of sentences. Then the relation \( \text{Prf}_\Gamma(x, y) \) expressing “\( x \) is the code of a derivation \( \pi \) of \( \Gamma_0 \Rightarrow \varphi \) for some finite \( \Gamma_0 \subseteq \Gamma \) and \( y \) is the Gödel number of \( \varphi \)” is primitive recursive.

**Proof.** Suppose “\( y \in \Gamma \)” is given by the primitive recursive predicate \( R_\Gamma(y) \). We have to show that \( \text{Prf}_\Gamma(x, y) \) which holds iff \( y \) is the Gödel number of a sentence \( \varphi \) and \( x \) is the code of an \( \text{LK-derivation} \) with end-sequent \( \Gamma_0 \Rightarrow \varphi \) is primitive recursive.

By the previous proposition, the property \( \text{Deriv}(x) \) which holds iff \( x \) is the code of a correct derivation \( \pi \) in \( \text{LK} \) is primitive recursive. If \( x \) is such a code, then \( \text{EndSequent}(x) \) is the code of the end-sequent of \( \pi \), and so \( (\text{EndSequent}(x))_0 \) is the code of the left side of the end sequent and \( (\text{EndSequent}(x))_1 \) the right side. So we can express “the right side of the end-sequent of \( \pi \) is \( \varphi \)” as \( \text{len}((\text{EndSequent}(x))_1) = 1 \land (\text{EndSequent}(x))_0 = x \). The left side of the end-sequent of \( \pi \) is of course automatically finite, we just have to express that every sentence in it is in \( \Gamma \). Thus we can define \( \text{Prf}_\Gamma(x, y) \) by

\[
\text{Prf}_\Gamma(x, y) \iff \text{Deriv}(x) \land
\begin{align*}
(\forall i < \text{len}((\text{EndSequent}(x))_0)) \ & R_\Gamma(((\text{EndSequent}(x))_0)_i) \land \\
\text{len}((\text{EndSequent}(x))_1) = 1 \land (\text{EndSequent}(x))_1_0 = y.
\end{align*}
\]

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**Bibliography**