## art.1 Derivations in LK

inc:art:plk:sec

In order to arithmetize derivations, we must represent derivations as numbers. Since derivations are trees of sequents where each inference carries also a label, a recursive representation is the most obvious approach: we represent a derivation as a tuple, the components of which are the end-sequent, the label, and the representations of the sub-derivations leading to the premises of the last inference. explanation

**Definition art.1.** If $\Gamma$ is a finite sequence of sentences, $\Gamma = \langle \varphi_1, \dots, \varphi_n \rangle$, then ${}^\#\Gamma^\# = \langle {}^\#\varphi_1{}^\#, \dots, {}^\#\varphi_n{}^\# \rangle$.

If $\Gamma \Rightarrow \Delta$ is a sequent, then a Gödel number of $\Gamma \Rightarrow \Delta$ is

$$ {}^\#\Gamma \Rightarrow \Delta^\# = \langle {}^\#\Gamma^\#, {}^\#\Delta^\# \rangle $$

If $\pi$ is a derivation in **LK**, then ${}^\#\pi^\#$ is

1. $\langle 0, {}^\#\Gamma \Rightarrow \Delta^\# \rangle$ if $\pi$ consists only of the initial sequent $\Gamma \Rightarrow \Delta$.

2. $\langle 1, {}^\#\Gamma \Rightarrow \Delta^\#, k, {}^\#\pi'^\# \rangle$ if $\pi$ ends in an inference with one premise, $k$ is given by the following table according to which rule was used in the last inference, and $\pi'$ is the immediate subproof ending in the premise of the last inference.

| | | | | | | |
|---|---|---|---|---|---|---|
| Rule: | WL | WR | CL | CR | XL | XR |
| $k$: | 1 | 2 | 3 | 4 | 5 | 6 |

| | | | | | |
|---|---|---|---|---|---|
| Rule: | $\neg$L | $\neg$R | $\land$L | $\lor$R | $\to$R |
| $k$: | 7 | 8 | 9 | 10 | 11 |

| | | | | | |
|---|---|---|---|---|---|
| Rule: | $\forall$L | $\forall$R | $\exists$L | $\exists$R | $=$ |
| $k$: | 12 | 13 | 14 | 15 | 16 |

3. $\langle 2, {}^\#\Gamma \Rightarrow \Delta^\#, k, {}^\#\pi'^\#, {}^\#\pi''^\# \rangle$ if $\pi$ ends in an inference with two premises, $k$ is given by the following table according to which rule was used in the last inference, and $\pi'$, $\pi''$ are the immediate subproof ending in the left and right premise of the last inference, respectively.

| | | | | |
|---|---|---|---|---|
| Rule: | Cut | $\land$R | $\lor$L | $\to$L |
| $k$: | 1 | 2 | 3 | 4 |

Having settled on a representation of derivations, we must also show that we can manipulate such derivations primite recursively, and express their essential properties and relations so. Some operations are simple: e.g., given a Gödel number $d$ of a derivation, $(s)_1$ gives us the Gödel number of its end-sequent. Some are much harder. We'll at least sketch how to do this. The goal is to show that the relation "$\pi$ is a derivation of $\varphi$ from $\Gamma$" is a primitive recursive relation of the Gödel numbers of $\pi$ and $\varphi$. explanation

inc:art:plk:prop:followsby

**Proposition art.2.** *The following relations are primitive recursive:*

1. *$\Gamma \Rightarrow \Delta$ is an initial sequent.*

2. *$\Gamma \Rightarrow \Delta$ follows from $\Gamma' \Rightarrow \Delta'$ (and $\Gamma'' \Rightarrow \Delta''$) by a rule of* **LK**.

3. *$\pi$ is a correct* **LK**-*derivation.*

*Proof.* We have to show that the corresponding relations between Gödel numbers of formulas, sequences of Gödel numbers of formulas (which code sequences of formulas), and Gödel numbers of sequents, are primitive recursive.

1. $\Gamma \Rightarrow \Delta$ is an initial sequent if either there is a sentence $\varphi$ such that $\Gamma \Rightarrow \Delta$ is $\varphi \Rightarrow \varphi$, or there is a term $t$ such that $\Gamma \Rightarrow \Delta$ is $\emptyset \Rightarrow t = t$. In terms of Gödel numbers, $\mathrm{InitSeq}(s)$ holds iff

$$(\exists x < s)\ (\mathrm{Sent}(x) \land s = \langle\langle x\rangle, \langle x\rangle\rangle) \lor$$
$$(\exists t < s)\ (\mathrm{Term}(t) \land s = \langle 0, \langle {}^{\#}{=}({}^{\#} \frown t \frown {}^{\#},{}^{\#} \frown t \frown {}^{\#}){}^{\#}\rangle\rangle).$$

2. Here we have to show that for each rule of inference $R$ the relation $\mathrm{FollowsBy}_R(s, s')$ which holds if $s$ and $s'$ are the Gödel numbers of conclusion and premise of a correct application of $R$ is primitive recursive. If $R$ has two premises, $\mathrm{FollowsBy}_R$ of course has three arguments.

   For instance, $\Gamma \Rightarrow \Delta$ follows correctly from $\Gamma' \Rightarrow \Delta'$ by $\exists$R iff $\Gamma = \Gamma'$ and there is a sequence of formulas $\Delta''$, a formula $\varphi$, a variable $x$ and a closed term $t$ such that $\Delta' = \Delta'', \varphi[t/x]$ and $\Delta = \Delta'', \exists x\, \varphi$. We just have to translate this into Gödel numbers. If $s = {}^{\#}\Gamma \Rightarrow \Delta^{\#}$ then $(s)_0 = {}^{\#}\Gamma^{\#}$ and $(s)_1 = {}^{\#}\Delta^{\#}$. So, $\mathrm{FollowsBy}_{\exists\mathrm{R}}(s, s')$ holds iff

$$\begin{aligned}
&(s)_0 = (s')_0 \land \\
&(\exists d < s)\ (\exists f < s)\ (\exists x < s)\ (\exists t < s')\ (\mathrm{Frm}(f) \land \mathrm{Var}(y) \land \mathrm{Term}(t) \land \\
&\qquad (s')_1 = d \frown \langle \mathrm{Subst}(f, t, x)\rangle \land \\
&\qquad (s)_1 = d \frown \langle \#(\exists) \frown y \frown f\rangle
\end{aligned}$$

   The individual lines express, respectively, "$\Gamma = \Gamma$," "there is a sequence ($\Delta''$) with Gödel number $d$, a formula ($\varphi$) with Gödel number $f$, a variable with Gödel number $x$, and a term with Gödel number $t$," "$\Delta' = \Delta'', \varphi[t/x]$," and "$\Delta = \Delta'', \exists x\, \varphi$". (Remember that ${}^{\#}\Delta^{\#}$ is the number of a sequence of Gödel numbers of formulas in $\Delta$.)

3. We first define a helper relation $\mathrm{hDeriv}(s, n)$ which holds if $s$ codes a correct derivation to at least $n$ inferences up from the end sequent. If $n = 0$ we let the relation be satisfied by default. Otherwise, $\mathrm{hDeriv}(s, n{+}1)$ iff either $s$ consists just of an initial sequent, or it ends in a correct inference

and the codes of the immediate subderivations satisfy $\mathrm{hDeriv}(s, n)$.

$$\begin{aligned}
\mathrm{hDeriv}(s, 0) &\Leftrightarrow \mathrm{true} \\
\mathrm{hDeriv}(s, n+1) &\Leftrightarrow \\
&((s)_0 = 0 \land \mathrm{InitialSeq}((s)_1)) \lor \\
&((s)_0 = 1 \land \\
&\quad ((s)_2 = 1 \land \mathrm{FollowsBy}_{\mathrm{CL}}((s)_1, ((s)_3)_1)) \lor \\
&\quad\quad \vdots \\
&\quad ((s)_2 = 16 \land \mathrm{FollowsBy}_{=}((s)_1, ((s)_3)_1)) \land \\
&\quad \mathrm{hDeriv}((s)_3, n)) \lor \\
&((s)_0 = 2 \land \\
&\quad ((s)_2 = 1 \land \mathrm{FollowsBy}_{\mathrm{Cut}}((s)_1, ((s)_3)_1), ((s)_4)_1)) \lor \\
&\quad\quad \vdots \\
&\quad ((s)_2 = 4 \land \mathrm{FollowsBy}_{\to\mathrm{L}}((s)_1, ((s)_3)_1), ((s)_4)_1)) \land \\
&\quad \mathrm{hDeriv}((s)_3, n) \land \mathrm{hDeriv}((s)_4, n))
\end{aligned}$$

This is a primitive recursive definition. If the number $n$ is large enough, e.g., larger than the maximum number of inferences between an initial sequent and the end sequent in $s$, it holds of $s$ iff $s$ is the Gödel number of a correct derivation. The number $s$ itself is larger than that maximum number of inferences. So we can now define $\mathrm{Deriv}(s)$ by $\mathrm{hDeriv}(s, s)$.

□

**Problem art.1.** Define the following relations as in Proposition art.2:

1. $\mathrm{FollowsBy}_{\land\mathrm{R}}(s, s', s'')$,
2. $\mathrm{FollowsBy}_{=}(s, s')$,
3. $\mathrm{FollowsBy}_{\forall\mathrm{R}}(s, s')$.

**Proposition art.3.** *Suppose $\Gamma$ is a primitive recursive set of sentences. Then the relation $\mathrm{Prf}_\Gamma(x, y)$ expressing "$x$ is the code of a derivation $\pi$ of $\Gamma_0 \Rightarrow \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$ and $x$ is the Gödel number of $\varphi$" is primitive recursive.*

*Proof.* Suppose "$y \in \Gamma$" is given by the primitive recursive predicate $R_\Gamma(y)$. We have to show that $\mathrm{Prf}_\Gamma(x, y)$ which holds iff $y$ is the Gödel number of a sentence $\varphi$ and $x$ is the code of an **LK**-derivation with end sequent $\Gamma_0 \Rightarrow \varphi$ is primitive recursive.

By the previous proposition, the property $\mathrm{Deriv}(x)$ which holds iff $x$ is the code of a correct derivation $\pi$ in **LK** is primitive recursive. If $x$ is such a code, then $(x)_1$ is the code of the end sequent of $\pi$, and so $((x)_1)_0$ is the code of the left side of the end sequent and $((x)_1)_1$ the right side. So we can express "the right side of the end sequent of $\pi$ is $\varphi$" as $\mathrm{len}(((x)_1)_1) = 1 \wedge (((x)_1)_1)_0 = x$. The left side of the end sequent of $\pi$ is of course automatically finite, we just have to express that every sentence in it is in $\Gamma$. Thus we can define $\mathrm{Prf}_\Gamma(x, y)$ by

$$\begin{aligned}\mathrm{Prf}_\Gamma(x, y) \Leftrightarrow\ & \mathrm{Sent}(y) \wedge \mathrm{Deriv}(x) \wedge \\ & (\forall i < \mathrm{len}(((x)_1)_0))\ R_\Gamma((((x)_1)_0)_i) \wedge \\ & \mathrm{len}(((x)_1)_1) = 1 \wedge (((x)_1)_1)_0 = x\end{aligned}$$

□

## Photo Credits

## Bibliography