

## art.1 Derivations in LK

inc:art:plk:  
sec In order to arithmetize **derivations**, we must represent **derivations** as numbers. explanation  
 Since **derivations** are trees of sequents where each inference carries also a label, a recursive representation is the most obvious approach: we represent a **derivation** as a tuple, the components of which are the end-sequent, the label, and the representations of the sub-**derivations** leading to the premises of the last inference.

**Definition art.1.** If  $\Gamma$  is a finite sequence of **sentences**,  $\Gamma = \langle \varphi_1, \dots, \varphi_n \rangle$ , then  $\# \Gamma^\# = \langle \# \varphi_1^\#, \dots, \# \varphi_n^\# \rangle$ .

If  $\Gamma \Rightarrow \Delta$  is a sequent, then a Gödel number of  $\Gamma \Rightarrow \Delta$  is

$$\# \Gamma \Rightarrow \Delta^\# = \langle \# \Gamma^\#, \# \Delta^\# \rangle$$

If  $\pi$  is a **derivation** in **LK**, then  $\# \pi^\#$  is defined as follows:

1. If  $\pi$  consists only of the initial sequent  $\Gamma \Rightarrow \Delta$ , then  $\# \pi^\#$  is

$$\langle 0, \# \Gamma \Rightarrow \Delta^\# \rangle.$$

2. If  $\pi$  ends in an inference with one or two premises, has  $\Gamma \Rightarrow \Delta$  as its conclusion, and  $\pi_1$  and  $\pi_2$  are the immediate subproof ending in the premise of the last inference, then  $\# \pi^\#$  is

$$\langle 1, \# \pi_1^\#, \# \Gamma \Rightarrow \Delta^\#, k \rangle \text{ or } \langle 2, \# \pi_1^\#, \# \pi_2^\#, \# \Gamma \Rightarrow \Delta^\#, k \rangle,$$

respectively, where  $k$  is given by the following table according to which rule was used in the last inference:

Rule:	WL	WR	CL	CR	XL	XR
$k$ :	1	2	3	4	5	6
Rule:	$\neg$ L	$\neg$ R	$\wedge$ L	$\wedge$ R	$\vee$ L	$\vee$ R
$k$ :	7	8	9	10	11	12
Rule:	$\rightarrow$ L	$\rightarrow$ R	$\forall$ L	$\forall$ R	$\exists$ L	$\exists$ R
$k$ :	13	14	15	16	17	18
Rule:	Cut	=				
$k$ :	19	20				

**Example art.2.** Consider the very simple **derivation**

$$\frac{\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi} \wedge L}{\Rightarrow (\varphi \wedge \psi) \rightarrow \varphi} \rightarrow R$$

The Gödel number of the initial sequent would be  $p_0 = \langle 0, \# \varphi \Rightarrow \varphi^\# \rangle$ . The Gödel number of the **derivation** ending in the conclusion of  $\wedge L$  would be  $p_1 = \langle 1, p_0, \# \varphi \wedge \psi \Rightarrow \varphi^\#, 9 \rangle$  (1 since  $\wedge L$  has one premise, the Gödel number of the conclusion  $\varphi \wedge \psi \Rightarrow \varphi$ , and 9 is the number coding  $\wedge L$ ). The Gödel number of the entire **derivation** then is  $\langle 1, p_1, \# \Rightarrow (\varphi \wedge \psi) \rightarrow \varphi^\#, 14 \rangle$ , i.e.,

$$\langle 1, \langle 1, \langle 0, \# \varphi \Rightarrow \varphi^\# \rangle, \# \varphi \wedge \psi \Rightarrow \varphi^\#, 9 \rangle, \# \Rightarrow (\varphi \wedge \psi) \rightarrow \varphi^\#, 14 \rangle.$$

explanation

Having settled on a representation of **derivations**, we must also show that we can manipulate such derivations primitive recursively, and express their essential properties and relations so. Some operations are simple: e.g., given a Gödel number  $p$  of a **derivation**,  $\text{EndSeq}(p) = (p)_{(p)_0+1}$  gives us the Gödel number of its end-sequent and  $\text{LastRule}(p) = (p)_{(p)_0+2}$  the code of its last rule. The property  $\text{Sequent}(s)$  defined by

$$\text{len}(s) = 2 \wedge (\forall i < \text{len}((s)_0) + \text{len}((s)_1)) \text{Sent}(((s)_0 \frown (s)_1)_i)$$

holds of  $s$  iff  $s$  is the Gödel number of a sequent consisting of **sentences**. Some are much harder. We'll at least sketch how to do this. The goal is to show that the relation “ $\pi$  is a **derivation** of  $\varphi$  from  $\Gamma$ ” is a primitive recursive relation of the Gödel numbers of  $\pi$  and  $\varphi$ .

**Proposition art.3.** *The property  $\text{Correct}(p)$  which holds iff the last inference in the **derivation**  $\pi$  with Gödel number  $p$  is correct, is primitive recursive.*

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prop:followsby

*Proof.*  $\Gamma \Rightarrow \Delta$  is an initial sequent if either there is a **sentence**  $\varphi$  such that  $\Gamma \Rightarrow \Delta$  is  $\varphi \Rightarrow \varphi$ , or there is a term  $t$  such that  $\Gamma \Rightarrow \Delta$  is  $\emptyset \Rightarrow t = t$ . In terms of Gödel numbers,  $\text{InitSeq}(s)$  holds iff

$$\begin{aligned} & (\exists x < s) (\text{Sent}(x) \wedge s = \langle \langle x \rangle, \langle x \rangle \rangle) \vee \\ & (\exists t < s) (\text{Term}(t) \wedge s = \langle 0, \langle \# = (\# \frown t \frown \#, \# \frown t \frown \#) \# \rangle \rangle). \end{aligned}$$

We also have to show that for each rule of inference  $R$  the relation  $\text{FollowsBy}_R(p)$  is primitive recursive, where  $\text{FollowsBy}_R(p)$  holds iff  $p$  is the Gödel number of **derivation**  $\pi$ , and the end-sequent of  $\pi$  follows by a correct application of  $R$  from the immediate sub-**derivations** of  $\pi$ .

A simple case is that of the  $\wedge R$  rule. If  $\pi$  ends in a correct  $\wedge R$  inference, it looks like this:

$$\frac{\begin{array}{c} \vdots \\ \vdots \pi_1 \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \pi_2 \\ \vdots \\ \Gamma \Rightarrow \Delta, \psi \end{array}}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge R$$

So, the last inference in the **derivation**  $\pi$  is a correct application of  $\wedge R$  iff there are sequences of **sentences**  $\Gamma$  and  $\Delta$  as well as two **sentences**  $\varphi$  and  $\psi$  such that the end-sequent of  $\pi_1$  is  $\Gamma \Rightarrow \Delta, \varphi$ , the end-sequent of  $\pi_2$  is  $\Gamma \Rightarrow \Delta, \psi$ ,

and the end-sequent of  $\pi$  is  $\Gamma \Rightarrow \Delta, \varphi \wedge \psi$ . We just have to translate this into Gödel numbers. If  $s = \# \Gamma \Rightarrow \Delta \#$  then  $(s)_0 = \# \Gamma \#$  and  $(s)_1 = \# \Delta \#$ . So,  $\text{FollowsBy}_{\wedge R}(p)$  holds iff

$$\begin{aligned} & (\exists g < p) (\exists d < p) (\exists a < p) (\exists b < p) \\ & \text{EndSequent}(p) = \langle g, d \frown \langle \#(\# \frown a \frown \# \wedge \# \frown b \frown \#) \# \rangle \rangle \wedge \\ & \text{EndSequent}((p)_1) = \langle g, d \frown \langle a \rangle \rangle \wedge \\ & \text{EndSequent}((p)_2) = \langle g, d \frown \langle b \rangle \rangle \wedge \\ & (p)_0 = 2 \wedge \text{LastRule}(p) = 10. \end{aligned}$$

The individual lines express, respectively, “there is a sequence ( $\Gamma$ ) with Gödel number  $g$ , there is a sequence ( $\Delta$ ) with Gödel number  $d$ , a formula ( $\varphi$ ) with Gödel number  $a$ , and a formula ( $\psi$ ) with Gödel number  $b$ ,” such that “the end-sequent of  $\pi$  is  $\Gamma \Rightarrow \Delta, \varphi \wedge \psi$ ,” “the end-sequent of  $\pi_1$  is  $\Gamma \Rightarrow \Delta, \varphi$ ,” “the end-sequent of  $\pi_2$  is  $\Gamma \Rightarrow \Delta, \psi$ ,” and “ $\pi$  has two immediate subderivations and the last inference rule is  $\wedge R$  (with number 10).”

The last inference in  $\pi$  is a correct application of  $\exists R$  iff there are sequences  $\Gamma$  and  $\Delta$ , a formula  $\varphi$ , a variable  $x$ , and a term  $t$ , such that the end-sequent of  $\pi$  is  $\Gamma \Rightarrow \Delta, \exists x \varphi$  and the end-sequent of  $\pi_1$  is  $\Gamma \Rightarrow \Delta, \varphi[t/x]$ . So in terms of Gödel numbers, we have  $\text{FollowsBy}_{\exists R}(p)$  iff

$$\begin{aligned} & (\exists g < p) (\exists d < p) (\exists a < p) (\exists x < p) (\exists t < p) \\ & \text{EndSequent}(p) = \langle g, d \frown \langle \# \exists \# \frown x \frown a \rangle \rangle \wedge \\ & \text{EndSequent}((p)_1) = \langle g, d \frown \langle \text{Subst}(a, t, x) \rangle \rangle \wedge \\ & (p)_0 = 1 \wedge \text{LastRule}(p) = 18. \end{aligned}$$

We then define  $\text{Correct}(p)$  as

$$\begin{aligned} & \text{Sequent}(\text{EndSequent}(p)) \wedge \\ & [(\text{LastRule}(p) = 1 \wedge \text{FollowsBy}_{\text{WL}}(p)) \vee \dots \vee \\ & (\text{LastRule}(p) = 20 \wedge \text{FollowsBy}_{=} (p)) \vee \\ & (p)_0 = 0 \wedge \text{InitialSeq}(\text{EndSequent}(p))] \end{aligned}$$

The first line ensures that the end-sequent of  $d$  is actually a sequent consisting of **sentences**. The last line covers the case where  $p$  is just an initial sequent.  $\square$

**Problem art.1.** Define the following properties as in **Proposition art.3**:

1.  $\text{FollowsBy}_{\text{Cut}}(p)$ ,
2.  $\text{FollowsBy}_{\rightarrow L}(p)$ ,
3.  $\text{FollowsBy}_{=} (p)$ ,
4.  $\text{FollowsBy}_{\vee R}(p)$ .

For the last one, you will have to also show that you can test primitive recursively if the last inference of the **derivation** with Gödel number  $p$  satisfies the eigenvariable condition, i.e., the eigenvariable  $a$  of the  $\forall R$  does not occur in the end-sequent.

**Proposition art.4.** *The relation  $\text{Deriv}(p)$  which holds if  $p$  is the Gödel number of a correct **derivation**  $\pi$ , is primitive recursive.* inc:art:plk:  
prop:deriv

*Proof.* A **derivation**  $\pi$  is correct if every one of its inferences is a correct application of a rule, i.e., if every one of its sub-**derivations** ends in a correct inference. So,  $\text{Deriv}(d)$  iff

$$(\forall i < \text{len}(\text{SubtreeSeq}(p))) \text{Correct}((\text{SubtreeSeq}(p))_i). \quad \square$$

**Proposition art.5.** *Suppose  $\Gamma$  is a primitive recursive set of **sentences**. Then the relation  $\text{Prf}_\Gamma(x, y)$  expressing “ $x$  is the code of a **derivation**  $\pi$  of  $\Gamma_0 \Rightarrow \varphi$  for some finite  $\Gamma_0 \subseteq \Gamma$  and  $y$  is the Gödel number of  $\varphi$ ” is primitive recursive.*

*Proof.* Suppose “ $y \in \Gamma$ ” is given by the primitive recursive predicate  $R_\Gamma(y)$ . We have to show that  $\text{Prf}_\Gamma(x, y)$  which holds iff  $y$  is the Gödel number of a sentence  $\varphi$  and  $x$  is the code of an **LK-derivation** with end-sequent  $\Gamma_0 \Rightarrow \varphi$  is primitive recursive.

By the previous proposition, the property  $\text{Deriv}(x)$  which holds iff  $x$  is the code of a correct derivation  $\pi$  in **LK** is primitive recursive. If  $x$  is such a code, then  $\text{EndSequent}(x)$  is the code of the end-sequent of  $\pi$ , and so  $(\text{EndSequent}(x))_0$  is the code of the left side of the end sequent and  $(\text{EndSequent}(x))_1$  the right side. So we can express “the right side of the end-sequent of  $\pi$  is  $\varphi$ ” as  $\text{len}((\text{EndSequent}(x))_1) = 1 \wedge ((\text{EndSequent}(x))_1)_0 = x$ . The left side of the end-sequent of  $\pi$  is of course automatically finite, we just have to express that every sentence in it is in  $\Gamma$ . Thus we can define  $\text{Prf}_\Gamma(x, y)$  by

$$\begin{aligned} \text{Prf}_\Gamma(x, y) \Leftrightarrow & \text{Deriv}(x) \wedge \\ & (\forall i < \text{len}((\text{EndSequent}(x))_0)) R_\Gamma(((\text{EndSequent}(x))_0)_i) \wedge \\ & \text{len}((\text{EndSequent}(x))_1) = 1 \wedge ((\text{EndSequent}(x))_1)_0 = y. \quad \square \end{aligned}$$

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## Bibliography