

## art.1 Derivations in LK

inc:art:plk:  
sec

In order to arithmetize **derivations**, we must represent **derivations** as numbers. Since **derivations** are trees of sequents where each inference carries also a label, a recursive representation is the most obvious approach: we represent a **derivation** as a tuple, the components of which are the end-sequent, the label, and the representations of the sub-**derivations** leading to the premises of the last inference.

explanation

**Definition art.1.** If  $\Gamma$  is a finite sequence of **sentences**,  $\Gamma = \langle \varphi_1, \dots, \varphi_n \rangle$ , then  $\# \Gamma \# = \langle \# \varphi_1 \#, \dots, \# \varphi_n \# \rangle$ .

If  $\Gamma \Rightarrow \Delta$  is a sequent, then a Gödel number of  $\Gamma \Rightarrow \Delta$  is

$$\# \Gamma \Rightarrow \Delta \# = \langle \# \Gamma \#, \# \Delta \# \rangle$$

If  $\pi$  is a **derivation** in **LK**, then  $\# \pi \#$  is

1.  $\langle 0, \# \Gamma \Rightarrow \Delta \# \rangle$  if  $\pi$  consists only of the initial sequent  $\Gamma \Rightarrow \Delta$ .
2.  $\langle 1, \# \Gamma \Rightarrow \Delta \#, k, \# \pi' \# \rangle$  if  $\pi$  ends in an inference with one premise,  $k$  is given by the following table according to which rule was used in the last inference, and  $\pi'$  is the immediate subproof ending in the premise of the last inference.

Rule:	WL	WR	CL	CR	XL	XR
$k$ :	1	2	3	4	5	6

Rule:	$\neg$ L	$\neg$ R	$\wedge$ L	$\vee$ R	$\rightarrow$ R
$k$ :	7	8	9	10	11

Rule:	$\forall$ L	$\forall$ R	$\exists$ L	$\exists$ R	=
$k$ :	12	13	14	15	16

3.  $\langle 2, \# \Gamma \Rightarrow \Delta \#, k, \# \pi' \#, \# \pi'' \# \rangle$  if  $\pi$  ends in an inference with two premises,  $k$  is given by the following table according to which rule was used in the last inference, and  $\pi'$ ,  $\pi''$  are the immediate subproof ending in the left and right premise of the last inference, respectively.

Rule:	Cut	$\wedge$ R	$\vee$ L	$\rightarrow$ L
$k$ :	1	2	3	4

Having settled on a representation of **derivations**, we must also show that we can manipulate such derivations primitive recursively, and express their essential properties and relations so. Some operations are simple: e.g., given a Gödel number  $d$  of a **derivation**,  $(s)_1$  gives us the Gödel number of its end-sequent. Some are much harder. We'll at least sketch how to do this. The goal is to show that the relation " $\pi$  is a **derivation** of  $\varphi$  from  $\Gamma$ " is a primitive recursive relation of the Gödel numbers of  $\pi$  and  $\varphi$ .

explanation

inc:art:plk:  
prop:followsby

**Proposition art.2.** *The following relations are primitive recursive:*

1.  $\Gamma \Rightarrow \Delta$  is an initial sequent.
2.  $\Gamma \Rightarrow \Delta$  follows from  $\Gamma' \Rightarrow \Delta'$  (and  $\Gamma'' \Rightarrow \Delta''$ ) by a rule of **LK**.
3.  $\pi$  is a correct **LK-derivation**.

*Proof.* We have to show that the corresponding relations between Gödel numbers of **formulas**, sequences of Gödel numbers of **formulas** (which code sequences of **formulas**), and Gödel numbers of sequents, are primitive recursive.

1.  $\Gamma \Rightarrow \Delta$  is an initial sequent if either there is a **sentence**  $\varphi$  such that  $\Gamma \Rightarrow \Delta$  is  $\varphi \Rightarrow \varphi$ , or there is a term  $t$  such that  $\Gamma \Rightarrow \Delta$  is  $\emptyset \Rightarrow t = t$ . In terms of Gödel numbers,  $\text{InitSeq}(s)$  holds iff

$$\begin{aligned} & (\exists x < s) (\text{Sent}(x) \wedge s = \langle \langle x \rangle, \langle x \rangle \rangle) \vee \\ & (\exists t < s) (\text{Term}(t) \wedge s = \langle 0, \langle \# = (\# \frown t \frown \#, \# \frown t \frown \#) \# \rangle \rangle). \end{aligned}$$

2. Here we have to show that for each rule of inference  $R$  the relation  $\text{FollowsBy}_R(s, s')$  which holds if  $s$  and  $s'$  are the Gödel numbers of conclusion and premise of a correct application of  $R$  is primitive recursive. If  $R$  has two premises,  $\text{FollowsBy}_R$  of course has three arguments.

For instance,  $\Gamma \Rightarrow \Delta$  follows correctly from  $\Gamma' \Rightarrow \Delta'$  by  $\exists R$  iff  $\Gamma = \Gamma'$  and there is a sequence of **formulas**  $\Delta''$ , a **formula**  $\varphi$ , a variable  $x$  and a closed term  $t$  such that  $\Delta' = \Delta'', \varphi[t/x]$  and  $\Delta = \Delta'', \exists x \varphi$ . We just have to translate this into Gödel numbers. If  $s = \# \Gamma \Rightarrow \Delta \#$  then  $(s)_0 = \# \Gamma \#$  and  $(s)_1 = \# \Delta \#$ . So,  $\text{FollowsBy}_{\exists R}(s, s')$  holds iff

$$\begin{aligned} & (s)_0 = (s')_0 \wedge \\ & (\exists d < s) (\exists f < s) (\exists x < s) (\exists t < s') (\text{Frm}(f) \wedge \text{Var}(y) \wedge \text{Term}(t) \wedge \\ & \quad (s')_1 = d \frown \langle \text{Subst}(f, t, x) \rangle \wedge \\ & \quad (s)_1 = d \frown \langle \#(\exists) \frown y \frown f \rangle \end{aligned}$$

The individual lines express, respectively, “ $\Gamma = \Gamma'$ ,” “there is a sequence  $(\Delta'')$  with Gödel number  $d$ , a **formula**  $(\varphi)$  with Gödel number  $f$ , a variable with Gödel number  $x$ , and a term with Gödel number  $t$ ,” “ $\Delta' = \Delta'', \varphi[t/x]$ ,” and “ $\Delta = \Delta'', \exists x \varphi$ ”. (Remember that  $\# \Delta \#$  is the number of a sequence of Gödel numbers of **formulas** in  $\Delta$ .)

3. We first define a helper relation  $\text{hDeriv}(s, n)$  which holds if  $s$  codes a correct derivation to at least  $n$  inferences up from the end sequent. If  $n = 0$  we let the relation be satisfied by default. Otherwise,  $\text{hDeriv}(s, n+1)$  iff either  $s$  consists just of an initial sequent, or it ends in a correct inference

and the codes of the immediate subderivations satisfy  $\text{hDeriv}(s, n)$ .

$$\begin{aligned}
& \text{hDeriv}(s, 0) \Leftrightarrow \text{true} \\
& \text{hDeriv}(s, n + 1) \Leftrightarrow \\
& \quad ((s)_0 = 0 \wedge \text{InitialSeq}((s)_1)) \vee \\
& \quad ((s)_0 = 1 \wedge \\
& \quad \quad ((s)_2 = 1 \wedge \text{FollowsBy}_{\text{CL}}((s)_1, ((s)_3)_1)) \vee \\
& \quad \quad \vdots \\
& \quad \quad ((s)_2 = 16 \wedge \text{FollowsBy}_{=}((s)_1, ((s)_3)_1)) \wedge \\
& \quad \quad \text{hDeriv}((s)_3, n)) \vee \\
& \quad ((s)_0 = 2 \wedge \\
& \quad \quad ((s)_2 = 1 \wedge \text{FollowsBy}_{\text{Cut}}((s)_1, ((s)_3)_1, ((s)_4)_1)) \vee \\
& \quad \quad \vdots \\
& \quad \quad ((s)_2 = 4 \wedge \text{FollowsBy}_{\rightarrow\text{L}}((s)_1, ((s)_3)_1, ((s)_4)_1)) \wedge \\
& \quad \quad \text{hDeriv}((s)_3, n) \wedge \text{hDeriv}((s)_4, n))
\end{aligned}$$

This is a primitive recursive definition. If the number  $n$  is large enough, e.g., larger than the maximum number of inferences between an initial sequent and the end sequent in  $s$ , it holds of  $s$  iff  $s$  is the Gödel number of a correct derivation. The number  $s$  itself is larger than that maximum number of inferences. So we can now define  $\text{Deriv}(s)$  by  $\text{hDeriv}(s, s)$ .

□

**Problem art.1.** Define the following relations as in [Proposition art.2](#):

1.  $\text{FollowsBy}_{\wedge\text{R}}(s, s', s'')$ ,
2.  $\text{FollowsBy}_{=} (s, s')$ ,
3.  $\text{FollowsBy}_{\vee\text{R}}(s, s')$ .

**Proposition art.3.** *Suppose  $\Gamma$  is a primitive recursive set of sentences. Then the relation  $\text{Prf}_{\Gamma}(x, y)$  expressing “ $x$  is the code of a derivation  $\pi$  of  $\Gamma_0 \Rightarrow \varphi$  for some finite  $\Gamma_0 \subseteq \Gamma$  and  $x$  is the Gödel number of  $\varphi$ ” is primitive recursive.*

*Proof.* Suppose “ $y \in \Gamma$ ” is given by the primitive recursive predicate  $R_{\Gamma}(y)$ . We have to show that  $\text{Prf}_{\Gamma}(x, y)$  which holds iff  $y$  is the Gödel number of a sentence  $\varphi$  and  $x$  is the code of an **LK**-derivation with end sequent  $\Gamma_0 \Rightarrow \varphi$  is primitive recursive.

By the previous proposition, the property  $\text{Deriv}(x)$  which holds iff  $x$  is the code of a correct derivation  $\pi$  in **LK** is primitive recursive. If  $x$  is such a code, then  $(x)_1$  is the code of the end sequent of  $\pi$ , and so  $((x)_1)_0$  is the code of the left side of the end sequent and  $((x)_1)_1$  the right side. So we can express “the

right side of the end sequent of  $\pi$  is  $\varphi$ " as  $\text{len}(((x)_1)_1) = 1 \wedge (((x)_1)_1)_0 = x$ . The left side of the end sequent of  $\pi$  is of course automatically finite, we just have to express that every sentence in it is in  $\Gamma$ . Thus we can define  $\text{Prf}_\Gamma(x, y)$  by

$$\begin{aligned} \text{Prf}_\Gamma(x, y) \Leftrightarrow & \text{Sent}(y) \wedge \text{Deriv}(x) \wedge \\ & (\forall i < \text{len}(((x)_1)_0)) R_\Gamma(((x)_1)_0)_i \wedge \\ & \text{len}(((x)_1)_1) = 1 \wedge (((x)_1)_1)_0 = x \end{aligned}$$

□

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## Bibliography