art.1 Axiomatic Derivations

inc:art:pax:sec In order to arithmetize axiomatic derivations, we must represent derivations as numbers. Since derivations are simply sequences of formulas, the obvious approach is to code every derivation as the code of the sequence of codes of formulas in it. explanation

Definition art.1. If δ is an axiomatic derivation consisting of formulas φ_1, ..., φ_n, then ${}^{\#}\delta^{\#}$ is

$$\langle {}^{\#}\varphi_1{}^{\#}, \dots, {}^{\#}\varphi_n{}^{\#} \rangle.$$

Example art.2. Consider the very simple derivation:

1. $\psi \to (\psi \lor \varphi)$
2. $(\psi \to (\psi \lor \varphi)) \to (\varphi \to (\psi \to (\psi \lor \varphi)))$
3. $\varphi \to (\psi \to (\psi \lor \varphi))$

The Gödel number of this derivation would be

$$\begin{aligned}&\langle\, {}^{\#}\psi \to (\psi \lor \varphi)^{\#},\\ &\quad {}^{\#}(\psi \to (\psi \lor \varphi)) \to (\varphi \to (\psi \to (\psi \lor \varphi)))^{\#},\\ &\quad {}^{\#}\varphi \to (\psi \to (\psi \lor \varphi))^{\#}\,\rangle.\end{aligned}$$

Having settled on a representation of derivations, we must also show that we can manipulate such derivations primitive recursively, and express their essential properties and relations so. Some operations are simple: e.g., given a Gödel number d of a derivation, $(d)_{\mathrm{len}(d)-1}$ gives us the Gödel number of its end-formula. Some are much harder. We'll at least sketch how to do this. The goal is to show that the relation "δ is a derivation of φ from Γ" is primitive recursive in the Gödel numbers of δ and φ. explanation

inc:art:pax:prop:followsby **Proposition art.3.** *The following relations are primitive recursive:*

1. *φ is an axiom.*
2. *The i-th line in δ is justified by modus ponens*
3. *The i-th line in δ is justified by* QR.
4. *δ is a correct derivation.*

Proof. We have to show that the corresponding relations between Gödel numbers of formulas and Gödel numbers of derivations are primitive recursive.

1. We have a given list of axiom schemas, and φ is an axiom if it is of the form given by one of these schemas. Since the list of schemas is finite, it suffices to show that we can test primitive recursively, for each axiom schema, if φ is of that form. For instance, consider the axiom schema

$$\psi \to (\chi \to \psi).$$

φ is an instance of this axiom schema if there are formulas ψ and χ such that we obtain φ when we concatenate '(' with ψ with '$\rightarrow$' with '(' with χ with '$\rightarrow$' with ψ and with '))'. We can test the corresponding property of the Gödel number n of φ, since concatenation of sequences is primitive recursive and the Gödel numbers of ψ and χ must be smaller than the Gödel number of φ, since when the relation holds, both ψ and χ are sub-formulas of φ. Hence, we can define:

$$\begin{aligned}\mathrm{IsAx}_{\psi\rightarrow(\chi\rightarrow\psi)}(n) \Leftrightarrow\ & (\exists b < n)\ (\exists c < n)\ (\mathrm{Sent}(b) \land \mathrm{Sent}(c) \land \\ & n = {}^{\#}({}^{\#} \frown b \frown {}^{\#}{\rightarrow}^{\#} \frown {}^{\#}({}^{\#} \frown c \frown {}^{\#}{\rightarrow}^{\#} \frown b \frown {}^{\#}))^{\#}).\end{aligned}$$

If we have such a definition for each axiom schema, their disjunction defines the property $\mathrm{IsAx}(n)$, "n is the Gödel number of an axiom."

2. The i-th line in δ is justified by modus ponens iff there are lines j and $k < i$ where the sentence on line j is some formula φ, the sentence on line k is $\varphi \rightarrow \psi$, and the sentence on line i is ψ.

$$\begin{aligned}\mathrm{MP}(d, i) \Leftrightarrow\ & (\exists j < i)\ (\exists k < i) \\ & (d)_k = {}^{\#}({}^{\#} \frown (d)_j \frown {}^{\#}{\rightarrow}^{\#} \frown (d)_i \frown {}^{\#})^{\#}\end{aligned}$$

Since bounded quantification, concatenation, and $=$ are primitive recursive, this defines a primitive recursive relation.

3. A line in δ is justified by QR if it is of the form $\psi \rightarrow \forall x\, \varphi(x)$, a preceding line is $\psi \rightarrow \varphi(c)$ for some constant symbol c, and c does on occur in ψ. This is the case iff

    a) there is a sentence ψ and
    b) a formula $\varphi(x)$ with a single variable x free so that
    c) line i contains $\psi \rightarrow \forall x\, \varphi(x)$
    d) some line $j < i$ contains $\psi \rightarrow \varphi[c/x]$ for a constant c
    e) which does not occur in ψ.

All of these can be tested primitive recursively, since the Gödel numbers of ψ, $\varphi(x)$, and x are less than the Gödel number of the formula on line i, and that of a less than the Gödel number of the formula on line j:

$$\begin{aligned}\mathrm{QR}_1(d, i) \Leftrightarrow\ & (\exists b < (d)_i)\ (\exists x < (d)_i)\ (\exists a < (d)_i)\ (\exists c < (d)_j)\ (\\ & \mathrm{Var}(x) \land \mathrm{Const}(c) \land \\ & (d)_i = {}^{\#}({}^{\#} \frown b \frown {}^{\#}{\rightarrow}^{\#} \frown {}^{\#}\forall^{\#} \frown x \frown a \frown {}^{\#})^{\#} \land \\ & (d)_j = {}^{\#}({}^{\#} \frown b \frown {}^{\#}{\rightarrow}^{\#} \frown \mathrm{Subst}(a, c, x) \frown {}^{\#})^{\#} \land \\ & \mathrm{Sent}(b) \land \mathrm{Sent}(\mathrm{Subst}(a, c, x)) \land (\forall k < \mathrm{len}(b))\ (b)_k \neq (c)_0)\end{aligned}$$

Here we assume that c and x are the Gödel numbers of the variable and constant considered as terms (i.e., not their symbol codes). We test that

x is the only free variable of $\varphi(x)$ by testing if $\varphi(x)[c/x]$ is a sentence, and ensure that c does not occur in ψ by requiring that every symbol of ψ is different from c.

We leave the other version of QR as an exercise.

4. d is the Gödel number of a correct derivation iff every line in it is an axiom, or justified by modus ponens or QR. Hence:

$$\mathrm{Deriv}(d) \Leftrightarrow (\forall i < \mathrm{len}(d))\ (\mathrm{IsAx}((d)_i) \lor \mathrm{MP}(d, i) \lor \mathrm{QR}(d, i)) \qquad \square$$

Problem art.1. Define the following relations as in Proposition art.3:

1. $\mathrm{IsAx}_{\varphi\to(\psi\to(\varphi\land\psi))}(n)$,
2. $\mathrm{IsAx}_{\forall x\,\varphi(x)\to\varphi(t)}(n)$,
3. $\mathrm{QR}_2(d, i)$ (for the other version of QR).

Proposition art.4. *Suppose Γ is a primitive recursive set of sentences. Then the relation $\mathrm{Prf}_\Gamma(x, y)$ expressing "x is the code of a derivation δ of φ from Γ and y is the Gödel number of φ" is primitive recursive.*

Proof. Suppose "$y \in \Gamma$" is given by the primitive recursive predicate $R_\Gamma(y)$. We have to show that the relation $\mathrm{Prf}_\Gamma(x, y)$ is primitive recursive, where $\mathrm{Prf}_\Gamma(x, y)$ holds iff y is the Gödel number of a sentence φ and x is the code of a derivation of φ from Γ.

By the previous proposition, the property $\mathrm{Deriv}(x)$ which holds iff x is the code of a correct derivation δ is primitive recursive. However, that definition did not take into account the set Γ as an additional way to justify lines in the derivation. Our primitive recursive test of whether a line is justified by QR also left out of consideration the requirement that the constant c is not allowed to occur in Γ. It is possible to amend our definition so that it takes into account Γ directly, but it is easier to use Deriv and the deduction theorem. $\Gamma \vdash \varphi$ iff there is some finite list of sentences $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\{\psi_1, \ldots, \psi_n\} \vdash \varphi$. And by the deduction theorem, this is the case if $\vdash (\psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots))$. Whether a sentence with Gödel number z is of this form can be tested primitive recursively. So, instead of considering x as the Gödel number of a derivation of the sentence with Gödel number y *from* Γ, we consider x as the Gödel number of a derivation of a nested conditional of the above form from $\emptyset$.

First, if we have a sequence of sentences, we can primitive recursively form the conditional with all these sentences as antecedents and given sentence as consequent:

$$\begin{aligned}
\mathrm{hCond}(s, y, 0) &= y \\
\mathrm{hCond}(s, y, n+1) &= {}^{\#}({}^{\#} \frown (s)_n \frown {}^{\#}{\to}^{\#} \frown \mathrm{Cond}(s, y, n) \frown {}^{\#}){}^{\#} \\
\mathrm{Cond}(s, y) &= \mathrm{hCond}(s, y, \mathrm{len}(s))
\end{aligned}$$

So we can define $\mathrm{Prf}_\Gamma(x, y)$ by

$$\begin{aligned}\mathrm{Prf}_\Gamma(x, y) \Leftrightarrow (\exists s < \mathrm{sequenceBound}(x, x))\ (\\ (x)_{\mathrm{len}(x)-1} = \mathrm{Cond}(s, y) \land \\ (\forall i < \mathrm{len}(s))\ (s)_i \in \Gamma \land \\ \mathrm{Deriv}(x)).\end{aligned}$$

The bound on s is given by considering that each $(s)_i$ is the Gödel number of a sub-formula of the last line of the derivation, i.e., is less than $(x)_{\mathrm{len}(x)-1}$. The number of antecedents $\psi \in \Gamma$, i.e., the length of s, is less than the length of the last line of x. □

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Bibliography