

art.1 Coding Terms

inc:art:trm:
sec A term is simply a certain kind of sequence of symbols: it is built up explanation inductively from constants and variables according to the formation rules for terms. Since sequences of symbols can be coded as numbers—using a coding scheme for the symbols plus a way to code sequences of numbers—assigning Gödel numbers to terms is not difficult. The challenge is rather to show that the property a number has if it is the Gödel number of a correctly formed term is computable, or in fact primitive recursive.

inc:art:trm:
prop:term-primrec **Proposition art.1.** *The relations $\text{Term}(x)$ and $\text{ClTerm}(x)$ which hold iff x is the Gödel number of a term or a closed term, respectively, are primitive recursive.*

Proof. A sequence of symbols s is a term iff there is a sequence $s_0, \dots, s_{k-1} = s$ of terms which records how the term s was formed from **constant symbols** and **variables** according to the formation rules for terms. To express that such a putative formation sequence follows the formation rules it has to be the case that, for each $i < k$, either

1. s_i is a variable v_j , or
2. s_i is a **constant symbol** c_j , or
3. s_i is built from n terms t_1, \dots, t_n occurring prior to place i using an n -place **function symbol** f_j^n .

To show that the corresponding relation on Gödel numbers is primitive recursive, we have to express this condition primitively, i.e., using primitive recursive functions, relations, and bounded quantification.

Suppose y is the number that codes the sequence s_0, \dots, s_{k-1} , i.e., $y = \langle \#s_0\#, \dots, \#s_{k-1}\# \rangle$. It codes a formation sequence for the term with Gödel number x iff for all $i < k$:

1. there is a j such that $(y)_i = \#v_j\#$, or
2. there is a j such that $(y)_i = \#c_j\#$, or
3. there is an n and a number $z = \langle z_1, \dots, z_n \rangle$ such that each z_l is equal to some $(y)_{i'}$ for $i' < i$ and

$$(y)_i = \#f_j^n(\# \frown \text{flatten}(z) \frown \#)\#,$$

and moreover $(y)_{k-1} = x$. The function $\text{flatten}(z)$ turns the sequence $\langle \#t_1\#, \dots, \#t_n\# \rangle$ into $\#t_1, \dots, t_n\#$ and is primitive recursive.

The indices j , n , the Gödel numbers z_l of the terms t_l , and the code z of the sequence $\langle z_1, \dots, z_n \rangle$, in (3) are all less than y . We can replace k above with $\text{len}(y)$. Hence we can express “ y is the code of a formation sequence of the

term with Gödel number x ” in a way that shows that this relation is primitive recursive.

We now just have to convince ourselves that there is a primitive recursive bound on y . But if x is the Gödel number of a term, it must have a formation sequence with at most $\text{len}(x)$ terms (since every term in the formation sequence of s must start at some place in s , and no two subterms can start at the same place). The Gödel number of each subterm of s is of course $\leq x$. Hence, there always is a formation sequence with code $\leq x^{\text{len}(x)}$.

For `CTerm`, simply leave out the clause for `variables`. □

Alternative proof of Proposition art.1. The inductive definition says that `constant symbols` and `variables` are terms, and if t_1, \dots, t_n are terms, then so is $f_j^n(t_1, \dots, t_n)$, for any n and j . So terms are formed in stages: `constant symbols` and `variables` at stage 0, terms involving one `function symbol` at stage 1, those involving at least two nested `function symbols` at stage 2, etc. Let’s say that a sequence of symbols s is a term of level l iff s can be formed by applying the inductive definition of terms l (or fewer) times, i.e., it “becomes” a term by stage l or before. So s is a term of level $l + 1$ iff

1. s is a variable v_j , or
2. s is a `constant symbol` c_j , or
3. s is built from n terms t_1, \dots, t_n of level l and an n -place `function symbol` f_j^n .

To show that the corresponding relation on Gödel numbers is primitive recursive, we have to express this condition primitive recursively, i.e., using primitive recursive functions, relations, and bounded quantification.

The number x is the Gödel number of a term s of level $l + 1$ iff

1. there is a j such that $x = \#v_j\#$, or
2. there is a j such that $x = \#c_j\#$, or
3. there is an n , a j , and a number $z = \langle z_1, \dots, z_n \rangle$ such that each z_i is the Gödel number of a term of level l and

$$x = \#f_j^n(\# \frown \text{flatten}(z) \frown \#)\#,$$

and moreover $(y)_{k-1} = x$.

The indices j , n , the Gödel numbers z_i of the terms t_i , and the code z of the sequence $\langle z_1, \dots, z_n \rangle$, in (3) are all less than x . So we get a primitive recursive definition by:

$$\begin{aligned} \text{ITerm}(x, 0) &= \text{Var}(x) \vee \text{Const}(x) \\ \text{ITerm}(x, l + 1) &= \text{Var}(x) \vee \text{Const}(x) \vee \\ &(\exists z < x) ((\forall i < \text{len}(z)) \text{ITerm}((z)_i, l) \wedge \\ &(\exists j < x) x = (\#f_j^{\text{len}(z)}(\# \frown \text{flatten}(z) \frown \#)\#)) \end{aligned}$$

We can now define $\text{Term}(x)$ by $\text{lTerm}(x, x)$, since the level of a term is always less than the Gödel number of the term. \square

Problem art.1. Show that the function $\text{flatten}(z)$, which turns the sequence $\langle \#t_1\#, \dots, \#t_n\# \rangle$ into $\#t_1, \dots, t_n\#$, is primitive recursive.

*inc:art.term:
prop:num-primrec*

Proposition art.2. *The function $\text{num}(n) = \#\bar{n}\#$ is primitive recursive.*

Proof. We define $\text{num}(n)$ by primitive recursion:

$$\begin{aligned}\text{num}(0) &= \#0\# \\ \text{num}(n+1) &= \#s(\# \frown \text{num}(n) \frown \#)\#.\end{aligned}$$

\square

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Bibliography