art.1 Coding Terms

inc:art:trm: sec

A term is simply a certain kind of sequence of symbols: it is built up inductively from constants and variables according to the formation rules for terms. Since sequences of symbols can be coded as numbers—using a coding scheme for the symbols plus a way to code sequences of numbers—assigning Gödel numbers to terms is not difficult. The challenge is rather to show that the property a number has if it is the Gödel number of a correctly formed term is computable, or in fact primitive recursive. explanation

inc:art:trm: prop:term-primrec

Proposition art.1. *The relations* $\mathrm{Term}(x)$ *and* $\mathrm{ClTerm}(x)$ *which hold iff* x *is the Gödel number of a term or a closed term, respectively, are primitive recursive.*

Proof. A sequence of symbols s is a term iff there is a sequence $s_0, \dots, s_{k-1} = s$ of terms which records how the term s was formed from constant symbols and variables according to the formation rules for terms. To express that such a putative formation sequence follows the formation rules it has to be the case that, for each $i < k$, either

1. s_i is a variable v_j, or
2. s_i is a constant symbol c_j, or
3. s_i is built from n terms $t_1, \dots, t_n$ occurring prior to place i using an n-place function symbol f_j^n.

To show that the corresponding relation on Gödel numbers is primitive recursive, we have to express this condition primitive recursively, i.e., using primitive recursive functions, relations, and bounded quantification.

Suppose y is the number that codes the sequence $s_0, \dots, s_{k-1}$, i.e., $y = \langle {}^{\#}s_0{}^{\#}, \dots, {}^{\#}s_k{}^{\#} \rangle$. It codes a formation sequence for the term with Gödel number x iff for all $i < k$:

1. there is a j such that $(y)_i = {}^{\#}v_j{}^{\#}$, or
2. there is a j such that $(y)_i = {}^{\#}c_j{}^{\#}$, or
3. there is an n and a number $z = \langle z_1, \dots, z_n \rangle$ such that each z_l is equal to some $(y)_{i'}$ for $i' < i$ and

$$(y)_i = {}^{\#}f_j^n({}^{\#} \frown \mathrm{flatten}(z) \frown {}^{\#}){}^{\#},$$

and moreover $(y)_{k-1} = x$. The function $\mathrm{flatten}(z)$ turns the sequence $\langle {}^{\#}t_1{}^{\#}, \dots, {}^{\#}t_n{}^{\#} \rangle$ into ${}^{\#}t_1, \dots, t_n{}^{\#}$ and is primitive recursive.

The indices j, n, the Gödel numbers z_l of the terms t_l, and the code z of the sequence $\langle z_1, \dots, z_n \rangle$, in (3) are all less than y. We can replace k above with $\mathrm{len}(y)$. Hence we can express "y is the code of a formation sequence of the term with Gödel number x" in a way that shows that this relation is primitive recursive.

We now just have to convince ourselves that there is a primitive recursive bound on y. But if x is the Gödel number of a term, it must have a formation sequence with at most $\mathrm{len}(x)$ terms (since every term in the formation sequence of s must start at some place in s, and no two subterms can start at the same place). The Gödel number of each subterm of s is of course $\leq x$. Hence, there always is a formation sequence with code $\leq x^{\mathrm{len}(x)}$.

For ClTerm, simply leave out the clause for variables. □

Alternative proof of Proposition art.1. The inductive definition says that constant symbols and variables are terms, and if $t_1, \ldots, t_n$ are terms, then so is $f^n_j(t_1, \ldots, t_n)$, for any n and j. So terms are formed in stages: constant symbols and variables at stage 0, terms involving one function symbol at stage 1, those involving at least two nested function symbols at stage 2, etc. Let's say that a sequence of symbols s is a term of level l iff s can be formed by applying the inductive definition of terms l (or fewer) times, i.e., it "becomes" a term by stage l or before. So s is a term of level $l+1$ iff

1. s is a variable v_j, or
2. s is a constant symbol c_j, or
3. s is built from n terms $t_1, \ldots, t_n$ of level l and an n-place function symbol f^n_j.

To show that the corresponding relation on Gödel numbers is primitive recursive, we have to express this condition primitive recursively, i.e., using primitive recursive functions, relations, and bounded quantification.

The number x is the Gödel number of a term s of level $l+1$ iff

1. there is a j such that $x = {}^{\#}v_j{}^{\#}$, or
2. there is a j such that $x = {}^{\#}c_j{}^{\#}$, or
3. there is an n, a j, and a number $z = \langle z_1, \ldots, z_n\rangle$ such that each z_i is the Gödel number of a term of level l and
$$x = {}^{\#}f^n_j({}^{\#} \frown \mathrm{flatten}(z) \frown {}^{\#}){}^{\#},$$

and moreover $(y)_{k-1} = x$.

The indices j, n, the Gödel numbers z_i of the terms t_i, and the code z of the sequence $\langle z_1, \ldots, z_n\rangle$, in (3) are all less than x. So we get a primitive recursive definition by:

$$\begin{aligned}
\mathrm{lTerm}(x, 0) &= \mathrm{Var}(x) \lor \mathrm{Const}(x)\\
\mathrm{lTerm}(x, l+1) &= \mathrm{Var}(x) \lor \mathrm{Const}(x) \lor\\
&\qquad (\exists z < x)\ ((\forall i < \mathrm{len}(z))\ \mathrm{lTerm}((z)_i, l) \land\\
&\qquad\quad (\exists j < x)\ x = ({}^{\#}f^{\mathrm{len}(z)}_j({}^{\#} \frown \mathrm{flatten}(z) \frown {}^{\#}){}^{\#}))
\end{aligned}$$

We can now define $\mathrm{Term}(x)$ by $\mathrm{lTerm}(x, x)$, since the level of a term is always less than the Gödel number of the term. $\square$

Problem art.1. Show that the function $\mathrm{flatten}(z)$, which turns the sequence $\langle {}^{\#}t_1{}^{\#}, \dots, {}^{\#}t_n{}^{\#} \rangle$ into ${}^{\#}t_1, \dots, t_n{}^{\#}$, is primitive recursive.

inc:art:trm: prop:num-primrec **Proposition art.2.** *The function* $\mathrm{num}(n) = {}^{\#}\overline{n}^{\#}$ *is primitive recursive.*

Proof. We define $\mathrm{num}(n)$ by primitive recursion:

$$\begin{aligned} \mathrm{num}(0) &= {}^{\#}\mathsf{o}^{\#} \\ \mathrm{num}(n+1) &= {}^{\#}\prime({}^{\#} \frown \mathrm{num}(n) \frown {}^{\#}){}^{\#}. \end{aligned}$$

$\square$

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Bibliography