The basic language $\mathcal{L}$ of first order logic makes use of the symbols

\[
\perp \; \neg \; \lor \; \land \; \to \; \forall \; \exists \; \; (\; )\; ,
\]

together with enumerable sets of variables and constant symbols, and enumerable sets of function symbols and predicate symbols of arbitrary arity. We can assign \emph{codes} to each of these symbols in such a way that every symbol is assigned a unique number as its code, and no two different symbols are assigned the same number. We know that this is possible since the set of all symbols is enumerable and so there is a bijection between it and the set of natural numbers. But we want to make sure that we can recover the symbol (as well as some information about it, e.g., the arity of a function symbol) from its code in a computable way. There are many possible ways of doing this, of course.

Here is one such way, which uses primitive recursive functions. (Recall that $\langle n_0, \ldots, n_k \rangle$ is the number coding the sequence of numbers $n_0$, \ldots, $n_k$.)

\begin{definition}
If $s$ is a symbol of $\mathcal{L}$, let the \emph{symbol code} $c_s$ be defined as follows:

1. If $s$ is among the logical symbols, $c_s$ is given by the following table:

\[
\begin{array}{cccc}
\perp & \neg & \lor & \land & \to & \forall \\
0,0 & 0,1 & 0,2 & 0,3 & 0,4 & 0,5 \\
\exists & = & ( & ) & ,
\end{array}
\]

\[
\begin{array}{cccc}
\forall\\n0,6 & 0,7 & 0,8 & 0,9 & 0,10
\end{array}
\]

2. If $s$ is the $i$-th variable $v_i$, then $c_s = \langle 1, i \rangle$.

3. If $s$ is the $i$-th constant symbol $c_i$, then $c_s = \langle 2, i \rangle$.

4. If $s$ is the $i$-th $n$-ary function symbol $f^n_i$, then $c_s = \langle 3, n, i \rangle$.

5. If $s$ is the $i$-th $n$-ary predicate symbol $P^n_i$, then $c_s = \langle 4, n, i \rangle$.
\end{definition}

\begin{proposition}
The following relations are primitive recursive:

1. $\text{Fn}(x,n)$ iff $x$ is the code of $f^n_i$ for some $i$, i.e., $x$ is the code of an $n$-ary function symbol.

2. $\text{Pred}(x,n)$ iff $x$ is the code of $P^n_i$ for some $i$ or $x$ is the code of $=$ and $n = 2$, i.e., $x$ is the code of an $n$-ary predicate symbol.
\end{proposition}

\begin{definition}
If $s_0, \ldots, s_{n-1}$ is a sequence of symbols, its \emph{Gödel number} is $(c_{s_0}, \ldots, c_{s_{n-1}})$.

Note that codes and Gödel numbers are different things. For instance, the variable $v_5$ has a code $c_{v_5} = \langle 1, 5 \rangle = 2^2 \cdot 3^5$. But the variable $v_5$ considered as a term is also a sequence of symbols (of length 1). The Gödel number $*v_5*$ of the term $v_5$ is $(c_{v_5}) = 2^{c_{v_5} + 1} = 2^{2^2 \cdot 3^5 + 1}$.
\end{definition}
Example art.4. Recall that if $k_0, \ldots, k_{n-1}$ is a sequence of numbers, then the code of the sequence $\langle k_0, \ldots, k_{n-1} \rangle$ in the power-of-primes coding is

$$2^{k_0+1} \cdot 3^{k_1+1} \cdots p_{n-1}^{k_{n-1}},$$

where $p_i$ is the $i$-th prime (starting with $p_0 = 2$). So for instance, the formula $v_0 = 0$, or, more explicitly, $=(v_0, c_0)$, has the Gödel number

$$\langle c_=, c_=(v_0, c_0, c_0, c) \rangle.$$

Here, $c_=$ is $\langle 0, 7 \rangle = 2^0+1 \cdot 3^7 = 1$, $c_{v_0}$ is $\langle 1, 0 \rangle = 2^1+1 \cdot 3^0+1$, etc. So $^* = (v_0, c_0)^*$ is

$$2^{c_+=1} \cdot 3^{c_+(v_0+1)} \cdot 5^{c_{v_0=0}} \cdot 7^{c_++1} \cdot 11^{c_{v_0=0}} \cdot 13^{c_{v_0=0}+1} =$$

$$2^{2^{1+3^{8+1}}} \cdot 3^{2^{7\cdot 3^{9+1}}} \cdot 5^{2^{2\cdot 3^{1+1}}} \cdot 7^{2^{1\cdot 3^{11+1}}} \cdot 11^{2^{3\cdot 3^{1+1}}} \cdot 13^{2^{1\cdot 3^{10+1}}} =$$

$$2^{13\cdot 123} \cdot 3^{3^{39\cdot 367}} \cdot 5^{13} \cdot 7^{3^{54\cdot 295}} \cdot 11^{2^{5}} \cdot 13^{118\cdot 099}.$$

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Bibliography