

set.1 Pathologies

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However, the definition of a *limit* turned out to allow for some rather “pathological” constructions.

Around the 1830s, Bolzano discovered a function which was *continuous everywhere*, but *differentiable nowhere*. (Unfortunately, Bolzano never published this; the idea was first encountered by mathematicians in 1872, thanks to Weierstrass’s independent discovery of the same idea.)¹ This was, to say the least, rather surprising. It is easy to find functions, such as $|x|$, which are continuous everywhere but not differentiable at a particular point. But a function which is continuous everywhere but differentiable *nowhere* is a very different beast. Consider, for a moment, how you might try to draw such a function. To ensure it is continuous, you must be able to draw it without ever removing your pen from the page; but to ensure it is differentiable nowhere, you would have to abruptly change the direction of your pen, constantly.

Further “pathologies” followed. In January 5 1874, Cantor wrote a letter to Dedekind, posing the problem:

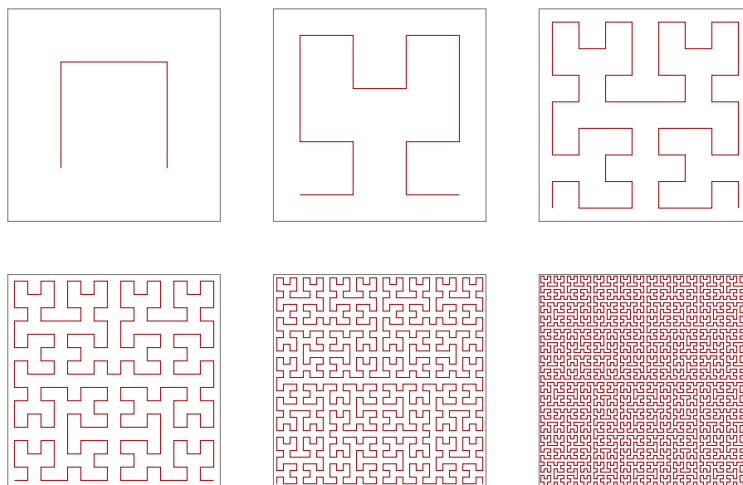
Can a surface (say a square including its boundary) be one-to-one correlated to a line (say a straight line including its endpoints) so that to every point of the surface there corresponds a point of the line, and conversely to every point of the line there corresponds a point of the surface?

It still seems to me at the moment that the answer to this question is very difficult—although here too one is so impelled to say *no* that one would like to hold the proof to be almost superfluous. [Quoted in [Gouvêa 2011](#)]

But, in 1877, Cantor proved that he had been wrong. In fact, a line and a square have exactly the same number of points. He wrote on 29 June 1877 to Dedekind “*je le vois, mais je ne le crois pas*”; that is, “I see it, but I don’t believe it”. In the “received history” of mathematics, this is often taken to indicate just how *literally incredible* these new results were to the mathematicians of the time. (The correspondence is presented in [Gouvêa \(2011\)](#), and we return to it in ??.) Cantor’s proof is outlined in ??.)

Inspired by Cantor’s result, Peano started to consider whether it might be possible to map a line *smoothly* onto a plane. This would be a *curve which fills space*. In 1890, Peano constructed just such a curve. This is truly counter-intuitive: Euclid had defined a line as “breadthless length” (Book I, Definition 2), but Peano had shown that, by curling up a line appropriately, its length can be turned into breadth. In 1891, Hilbert described a slightly more intuitive space-filling curve, together with some pictures illustrating it. The curve is constructed in sequence, and here are the first six stages of the construction:

¹The history is documented in extremely thorough footnotes to the Wikipedia article on the Weierstrass function.



In the limit—a notion which had, by now, received rigorous definition—the entire square is filled in solid red. And, in passing, Hilbert’s curve is continuous everywhere but differentiable nowhere; intuitively because, in the infinite limit, the function abruptly changes direction at every moment. (We will outline Hilbert’s construction in more detail in ??.)

For better or worse, these “pathological” geometric constructions were treated as a reason to doubt appeals to geometric intuition. They became something approaching *propaganda* for a new way of doing mathematics, which would culminate in set theory. In the later myth-building of the subject, it was repeated, often, that these results were both perfectly rigorous and perfectly shocking. They therefore served a dual purpose: as a warning against relying upon geometric intuition, and as a demonstration of the fertility of new ways of thinking.

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Bibliography

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- Hilbert, David. 1891. Über die stetige Abbildung einer Linie auf ein Flächenstück. *Mathematische Annalen* 38(3): 459–460.
- Peano, Giuseppe. 1890. Sur une courbe, qui remplit toute une aire plane. *Mathematische Annalen* 36(1): 157–60.