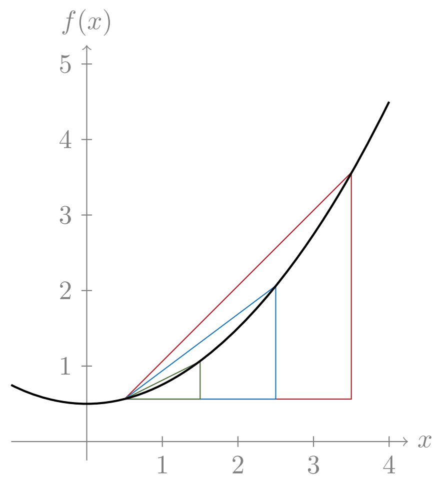
set.1 Infinitesimals and Differentiation

his:set:infinitesimals:sec

Newton and Leibniz discovered the calculus (independently) at the end of the 17th century. A particularly important application of the calculus was *differentiation*. Roughly speaking, differentiation aims to give a notion of the "rate of change", or gradient, of a function at a point.

Here is a vivid way to illustrate the idea. Consider the function $f(x) = {}^{x^2}\!/_4 + {}^1\!/_2$, depicted in black below:



Suppose we want to find the gradient of the function at $c = {}^1\!/_2$. We start by drawing a triangle whose hypotenuse approximates the gradient at that point, perhaps the red triangle above. When β is the base length of our triangle, its height is $f({}^1\!/_2 + \beta) - f({}^1\!/_2)$, so that the gradient of the hypotenuse is:

$$\frac{f({}^1\!/_2 + \beta) - f({}^1\!/_2)}{\beta}.$$

So the gradient of our red triangle, with base length 3, is exactly 1. The hypotenuse of a smaller triangle, the blue triangle with base length 2, gives a better approximation; its gradient is ${}^3\!/_4$. A yet smaller triangle, the green triangle with base length 1, gives a yet better approximation; with gradient ${}^1\!/_2$.

Ever-smaller triangles give us ever-better approximations. So we might say something like this: the hypotenuse of a triangle with an *infinitesimal* base length gives us the gradient at $c = {}^1\!/_2$ itself. In this way, we would obtain a formula for the (first) derivative of the function f at the point c:

$$f'(c) = \frac{f(c + \beta) - f(c)}{\beta} \text{ where } \beta \text{ is infinitesimal.}$$

And, roughly, this is what Newton and Leibniz said.

However, since they have said this, we must ask them: what is an *infinitesimal*? A serious dilemma arises. If $\beta = 0$, then f' is ill-defined, for it involves dividing by 0. But if $\beta > 0$, then we just get an *approximation* to the gradient, and not the gradient itself.

This is not an anachronistic concern. Here is Berkeley, criticizing Newton's followers:

> I admit that signs may be made to denote either any thing or nothing: and consequently that in the original notation $c + \beta$, β might have signified either an increment or nothing. But then which of these soever you make it signify, you must argue consistently with such its signification, and not proceed upon a double meaning: Which to do were a manifest sophism. (Berkeley 1734, §XIII, variables changed to match preceding text)

To defend the infinitesimal calculus against Berkeley, one might reply that the talk of "infinitesimals" is merely figurative. One might say that, so long as we take a *really small* triangle, we will get a *good enough* approximation to the tangent. Berkeley had a reply to this too: whilst that might be good enough for engineering, it undermines the *status* of mathematics, for

> we are told that *in rebus mathematicis errores quàm minimi non sunt contemnendi*. [In the case of mathematics, the smallest errors are not to be neglected.] (Berkeley, 1734, §IX)

The italicised passage is a near-verbatim quote from Newton's own *Quadrature of Curves* (1704).

Berkeley's philosophical objections are deeply incisive. Nevertheless, the calculus was a massively successful enterprise, and mathematicians continued to use it without falling into error.

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Bibliography

Berkeley, George. 1734. *The Analyst; or, a Discourse Adressed to an Infidel Mathematician.*