set.1 Cantor on the Line and the Plane

Some of the circumstances surrounding the proof of Schröder-Bernstein tie in with the history we discussed in ???. Recall that, in 1877, Cantor proved that there are exactly as many points on a square as on one of its sides. Here, we will present his (first attempted) proof.

Let \( L \) be the unit line, i.e., the set of points \([0, 1]\). Let \( S \) be the unit square, i.e., the set of points \( L \times L \). In these terms, Cantor proved that \( L \approx S \). He wrote a note to Dedekind, essentially containing the following argument.

**Theorem set.1.** \( L \approx S \)

**Proof:** first part. Fix \( a, b \in L \). Write them in binary notation, so that we have infinite sequences of 0s and 1s, \( a_1, a_2, \ldots, \) and \( b_1, b_2, \ldots, \) such that:

\[
a = 0.a_1a_2a_3a_4\ldots \\
b = 0.b_1b_2b_3b_4\ldots
\]

Now consider the function \( f: S \to L \) given by

\[
f(a, b) = 0.a_1b_1a_2b_2a_3b_3a_4b_4\ldots
\]

Now \( f \) is an injection, since if \( f(a, b) = f(c, d) \), then \( a_n = c_n \) and \( b_n = d_n \) for all \( n \in \mathbb{N} \), so that \( a = c \) and \( b = d \).

Unfortunately, as Dedekind pointed out to Cantor, this does not answer the original question. Consider \( 0.\dot{1}\dot{0} = 0.1010010010\ldots \). We need that \( f(a, b) = 0.\dot{1}\dot{0} \), where:

\[
a = 0.\dot{1}\dot{1} = 0.111111\ldots \\
b = 0
\]

But \( a = 0.\dot{1}\dot{1} = 1 \). So, when we say “write \( a \) and \( b \) in binary notation”, we have to choose which notation to use; and, since \( f \) is to be a function, we can use only one of the two possible notations. But if, for example, we use the simple notation, and write \( a \) as “1.000\ldots”, then we have no pair \( \langle a, b \rangle \) such that \( f(a, b) = 0.\dot{1}\dot{0} \).

To summarise: Dedekind pointed out that, given the possibility of certain recurring decimal expansions, Cantor’s function \( f \) is an injection but not a surjection. So Cantor has shown only that \( S \preceq L \) and not that \( S \approx L \).

Cantor wrote back to Dedekind almost immediately, essentially suggesting that the proof could be completed as follows:

**Proof: completed.** So, we have shown that \( S \preceq L \). But there is obviously an injection from \( L \) to \( S \): just lay the line flat along one side of the square. So \( L \preceq S \) and \( S \preceq L \). By Schröder–Bernstein (??), \( L \approx S \).
But of course, Cantor could not complete the last line in these terms, for
the Schröder-Bernstein Theorem was not yet proved. Indeed, although Cantor
would subsequently formulate this as a general conjecture, it was not satisfac-
torily proved until 1897. (And so, later in 1877, Cantor offered a different proof
of Theorem set.1, which did not go via Schröder–Bernstein.)

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Bibliography