set.1 Cantor on the Line and the Plane

Some of the circumstances surrounding the proof of Schröder-Bernstein tie in with the history we discussed in ???. Recall that, in 1877, Cantor proved that there are exactly as many points on a square as on one of its sides. Here, we will present his (first attempted) proof.

Let $L$ be the unit line, i.e., the set of points $[0, 1]$. Let $S$ be the unit square, i.e., the set of points $L \times L$. In these terms, Cantor proved that $L \approx S$. He wrote a note to Dedekind, essentially containing the following argument.

**Theorem set.1.** $L \approx S$

**Proof:**

Fix $a, b \in L$. Write them in binary notation, so that we have infinite sequences of 0s and 1s, $a_1, a_2, \ldots$, and $b_1, b_2, \ldots$, such that:

- $a = 0.a_1a_2a_3a_4\ldots$
- $b = 0.b_1b_2b_3b_4\ldots$

Now consider the function $f : S \to L$ given by

$$f(a, b) = 0.a_1b_1a_2b_2a_3b_3a_4b_4\ldots$$

Now $f$ is an injection, since if $f(a, b) = f(c, d)$, then $a_n = c_n$ and $b_n = d_n$ for all $n \in \mathbb{N}$, so that $a = c$ and $b = d$. \hfill $\Box$

Unfortunately, as Dedekind pointed out to Cantor, this does not answer the original question. Consider $0.\dot{1}\dot{0} = 0.1010101010\ldots$. We need that $f(a, b) = 0.\dot{1}\dot{0}$, where:

- $a = 0.\dot{1}\dot{1} = 0.111111\ldots$
- $b = 0$

But $a = 0.\dot{1}\dot{1} = 1$. So, when we say “write $a$ and $b$ in binary notation”, we have to choose which notation to use; and, since $f$ is to be a function, we can use only one of the two possible notations. But if, for example, we use the simple notation, and write $a$ as “1.000\ldots”, then we have no pair $\langle a, b \rangle$ such that $f(a, b) = 0.\dot{1}\dot{0}$.

To summarise: Dedekind pointed out that, given the possibility of certain recurring decimal expansions, Cantor’s function $f$ is an injection but not a surjection. So Cantor has shown only that $S \preceq L$ and not that $S \approx L$.

Cantor wrote back to Dedekind almost immediately, essentially suggesting that the proof could be completed as follows:

**Proof: completed.** So, we have shown that $S \preceq L$. But there is obviously an injection from $L$ to $S$: just lay the line flat along one side of the square. So $L \preceq S$ and $S \preceq L$. By Schröder–Bernstein (??), $L \approx S$. \hfill $\Box$
But of course, Cantor could not complete the last line in these terms, for the Schröder-Bernstein Theorem was not yet proved. Indeed, although Cantor would subsequently formulate this as a general conjecture, it was not satisfactorily proved until 1897. (And so, later in 1877, Cantor offered a different proof of Theorem set.1, which did not go via Schröder–Bernstein.)