set.1 Cantor on the Line and the Plane

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Some of the circumstances surrounding the proof of Schröder-Bernstein tie in with the history we discussed in **??**. Recall that, in 1877, Cantor proved that there are exactly as many points on a square as on one of its sides. Here, we will present his (first attempted) proof.

Let L be the unit line, i.e., the set of points $[0, 1]$. Let S be the unit square, i.e., the set of points $\mathrm{L} \times \mathrm{L}$. In these terms, Cantor proved that $\mathrm{L} \approx \mathrm{S}$. He wrote a note to Dedekind, essentially containing the following argument.

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Theorem set.1. $L \approx S$

Proof: first part. Fix $a, b \in \mathrm{L}$. Write them in binary notation, so that we have infinite sequences of 0s and 1s, a_1, a_2, ..., and b_1, b_2, ..., such that:

$$a = 0.a_1a_2a_3a_4\ldots$$
$$b = 0.b_1b_2b_3b_4\ldots$$

Now consider the function $f\colon \mathrm{S} \to \mathrm{L}$ given by

$$f(a, b) = 0.a_1b_1a_2b_2a_3b_3a_4b_4\ldots$$

Now f is an injection, since if $f(a, b) = f(c, d)$, then $a_n = c_n$ and $b_n = d_n$ for all $n \in \mathbb{N}$, so that $a = c$ and $b = d$. □

Unfortunately, as Dedekind pointed out to Cantor, this does not answer the original question. Consider $0.\dot{1}\dot{0} = 0.1010101010\ldots$. We need that $f(a, b) = 0.\dot{1}\dot{0}$, where:

$$a = 0.\dot{1}\dot{1} = 0.111111\ldots$$
$$b = 0$$

But $a = 0.\dot{1}\dot{1} = 1$. So, when we say "write a and b in binary notation", we have to choose *which* notation to use; and, since f is to be a *function*, we can use only *one* of the two possible notations. But if, for example, we use the simple notation, and write a as "$1.000\ldots$", then we have no pair $\langle a, b\rangle$ such that $f(a, b) = 0.\dot{1}\dot{0}$.

To summarise: Dedekind pointed out that, given the possibility of certain recurring decimal expansions, Cantor's function f is an injection but *not* a surjection. So Cantor has shown only that $\mathrm{S} \preceq \mathrm{L}$ and *not* that $\mathrm{S} \approx \mathrm{L}$.

Cantor wrote back to Dedekind almost immediately, essentially suggesting that the proof could be completed as follows:

Proof: completed. So, we have shown that $\mathrm{S} \preceq \mathrm{L}$. But there is obviously an injection from L to S: just lay the line flat along one side of the square. So $\mathrm{L} \preceq \mathrm{S}$ and $\mathrm{S} \preceq \mathrm{L}$. By Schröder–Bernstein (**??**), $\mathrm{L} \approx \mathrm{S}$. □

But of course, Cantor could not complete the last line in these terms, for the Schröder-Bernstein Theorem was not yet proved. Indeed, although Cantor would subsequently formulate this as a general conjecture, it was not satisfactorily proved until 1897. (And so, later in 1877, Cantor offered a different proof of Theorem set.1, which did not go via Schröder–Bernstein.)

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Bibliography