

## set.1 Cantor on the Line and the Plane

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sec

Some of the circumstances surrounding the proof of Schröder-Bernstein tie in with the history we discussed in ???. Recall that, in 1877, Cantor proved that there are exactly as many points on a square as on one of its sides. Here, we will present his (first attempted) proof.

Let  $L$  be the unit line, i.e., the set of points  $[0, 1]$ . Let  $S$  be the unit square, i.e., the set of points  $L \times L$ . In these terms, Cantor proved that  $L \approx S$ . He wrote a note to Dedekind, essentially containing the following argument.

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**Theorem set.1.**  $L \approx S$

*Proof: first part..* Fix  $a, b \in L$ . Write them in binary notation, so that we have infinite sequences of 0s and 1s,  $a_1, a_2, \dots$ , and  $b_1, b_2, \dots$ , such that:

$$\begin{aligned} a &= 0.a_1a_2a_3a_4\dots \\ b &= 0.b_1b_2b_3b_4\dots \end{aligned}$$

Now consider the function  $f: S \rightarrow L$  given by

$$f(a, b) = 0.a_1b_1a_2b_2a_3b_3a_4b_4\dots$$

Now  $f$  is an **injection**, since if  $f(a, b) = f(c, d)$ , then  $a_n = c_n$  and  $b_n = d_n$  for all  $n \in \mathbb{N}$ , so that  $a = c$  and  $b = d$ .  $\square$

Unfortunately, as Dedekind pointed out to Cantor, this does not answer the original question. Consider  $0.\dot{1}0 = 0.10101010\dots$ . We need that  $f(a, b) = 0.\dot{1}0$ , where:

$$\begin{aligned} a &= 0.\dot{1}1 = 0.111111\dots \\ b &= 0 \end{aligned}$$

But  $a = 0.\dot{1}1 = 1$ . So, when we say “write  $a$  and  $b$  in binary notation”, we have to choose *which* notation to use; and, since  $f$  is to be a *function*, we can use only *one* of the two possible notations. But if, for example, we use the simple notation, and write  $a$  as “1.000...”, then we have no pair  $\langle a, b \rangle$  such that  $f(a, b) = 0.\dot{1}0$ .

To summarise: Dedekind pointed out that, given the possibility of certain recurring decimal expansions, Cantor’s function  $f$  is an **injection** but *not* a **surjection**. So Cantor has shown only that  $S \preceq L$  and *not* that  $S \approx L$ .

Cantor wrote back to Dedekind almost immediately, essentially suggesting that the proof could be completed as follows:

*Proof: completed..* So, we have shown that  $S \preceq L$ . But there is obviously an **injection** from  $L$  to  $S$ : just lay the line flat along one side of the square. So  $L \preceq S$  and  $S \preceq L$ . By Schröder-Bernstein (??),  $L \approx S$ .  $\square$

But of course, Cantor could not complete the last line in these terms, for the Schröder-Bernstein Theorem was not yet proved. Indeed, although Cantor would subsequently formulate this as a general conjecture, it was not satisfactorily proved until 1897. (And so, later in 1877, Cantor offered a different proof of [Theorem set.1](#), which did not go via Schröder–Bernstein.)

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## Bibliography