

# Chapter udf

## Tableaux

This chapter presents a signed analytic tableaux system.  
To include or exclude material relevant to natural deduction as a proof system, use the “prfTab” tag.

### tab.1 Rules and Tableaux

fol:tab:rul:  
sec A **tableau** is a systematic survey of the possible ways a **sentence** can be true or false in a **structure**. The building blocks of a tableau are **signed formulas: sentences** plus a truth value “sign,” either  $\mathbb{T}$  or  $\mathbb{F}$ . These signed **formulas** are arranged in a (downward growing) tree.

**Definition tab.1.** A *signed formula* is a pair consisting of a truth value and a **sentence**, i.e., either:

$$\mathbb{T}\varphi \text{ or } \mathbb{F}\varphi.$$

Intuitively, we might read  $\mathbb{T}\varphi$  as “ $\varphi$  might be true” and  $\mathbb{F}\varphi$  as “ $\varphi$  might be false” (in some **structure**).

Each **signed formula** in the tree is either an *assumption* (which are listed at the very top of the tree), or it is obtained from a **signed formula** above it by one of a number of rules of inference. There are two rules for each possible **main operator** of the preceding **formula**, one for the case when the sign is  $\mathbb{T}$ , and one for the case where the sign is  $\mathbb{F}$ . Some rules allow the tree to branch, and some only add **signed formulas** to the branch. A rule may be (and often must be) applied not to the immediately preceding **signed formula**, but to any **signed formula** in the branch from the root to the place the rule is applied.

A branch is *closed* when it contains both  $\mathbb{T}\varphi$  and  $\mathbb{F}\varphi$ . A closed **tableau** is one where every branch is closed. Under the intuitive interpretation, any branch describes a joint possibility, but  $\mathbb{T}\varphi$  and  $\mathbb{F}\varphi$  are not jointly possible. In other words, if a branch is closed, the possibility it describes has been ruled out. In particular, that means that a closed **tableau** rules out all possibilities

of simultaneously making every assumption of the form  $\mathbb{T}\varphi$  true and every assumption of the form  $\mathbb{F}\varphi$  false.

A closed **tableau for  $\varphi$**  is a closed **tableau** with root  $\mathbb{F}\varphi$ . If such a closed **tableau** exists, all possibilities for  $\varphi$  being false have been ruled out; i.e.,  $\varphi$  must be true in every **structure**.

## tab.2 Propositional Rules

fol:tab:prl:  
sec

### Rules for $\neg$

$$\frac{\mathbb{T}\neg\varphi}{\mathbb{F}\varphi} \neg\mathbb{T} \qquad \frac{\mathbb{F}\neg\varphi}{\mathbb{T}\varphi} \neg\mathbb{F}$$

### Rules for $\wedge$

$$\frac{\mathbb{T}\varphi \wedge \psi}{\mathbb{T}\varphi \quad \mathbb{T}\psi} \wedge\mathbb{T} \qquad \frac{\mathbb{F}\varphi \wedge \psi}{\mathbb{F}\varphi \quad | \quad \mathbb{F}\psi} \wedge\mathbb{F}$$

### Rules for $\vee$

$$\frac{\mathbb{T}\varphi \vee \psi}{\mathbb{T}\varphi \quad | \quad \mathbb{T}\psi} \vee\mathbb{T} \qquad \frac{\mathbb{F}\varphi \vee \psi}{\mathbb{F}\varphi \quad \mathbb{F}\psi} \vee\mathbb{F}$$

### Rules for $\rightarrow$

$$\frac{\mathbb{T}\varphi \rightarrow \psi}{\mathbb{F}\varphi \quad | \quad \mathbb{T}\psi} \rightarrow\mathbb{T} \qquad \frac{\mathbb{F}\varphi \rightarrow \psi}{\mathbb{T}\varphi \quad \mathbb{F}\psi} \rightarrow\mathbb{F}$$

### The Cut Rule

$$\frac{\mathbb{T}\varphi \quad | \quad \mathbb{F}\varphi}{\text{Cut}}$$

The Cut rule is not applied “to” a previous **signed formula**; rather, it allows every branch in a **tableau** to be split in two, one branch containing  $\mathbb{T}\varphi$ , the other  $\mathbb{F}\varphi$ . It is not necessary—any set of **signed formulas** with a closed **tableau** has one not using Cut—but it allows us to combine **tableaux** in a convenient way.

### tab.3 Quantifier Rules

fol:tab:qrl:  
sec

#### Rules for $\forall$

$$\frac{\mathbb{T}\forall x\varphi(x)}{\mathbb{T}\varphi(t)} \forall\mathbb{T} \qquad \frac{\mathbb{F}\forall x\varphi(x)}{\mathbb{F}\varphi(a)} \forall\mathbb{F}$$

In  $\forall\mathbb{T}$ ,  $t$  is a closed term (i.e., one without variables). In  $\forall\mathbb{F}$ ,  $a$  is a **constant symbol** which must not occur anywhere in the branch above  $\forall\mathbb{F}$  rule. We call  $a$  the *eigenvariable* of the  $\forall\mathbb{F}$  inference.

#### Rules for $\exists$

$$\frac{\mathbb{T}\exists x\varphi(x)}{\mathbb{T}\varphi(a)} \exists\mathbb{T} \qquad \frac{\mathbb{F}\exists x\varphi(x)}{\mathbb{F}\varphi(t)} \exists\mathbb{F}$$

Again,  $t$  is a closed term, and  $a$  is a **constant symbol** which does not occur in the branch above the  $\exists\mathbb{F}$  rule. We call  $a$  the *eigenvariable* of the  $\exists\mathbb{F}$  inference.

The condition that an eigenvariable not occur in the branch above the  $\forall\mathbb{F}$  or  $\exists\mathbb{T}$  inference is called the *eigenvariable condition*.

We use the term “eigenvariable” even though  $a$  in the above rules is a **constant symbol**. This has historical reasons. [explanation](#)

In  $\forall\mathbb{T}$  and  $\exists\mathbb{F}$  there are no restrictions on the term  $t$ . On the other hand, in the  $\exists\mathbb{T}$  and  $\forall\mathbb{F}$  rules, the eigenvariable condition requires that the **constant symbol**  $a$  does not occur anywhere in the branches above the respective inference. It is necessary to ensure that the system is sound. Without this condition, the following would be a closed **tableau** for  $\exists x\varphi(x) \rightarrow \forall x\varphi(x)$ :

1.	$\mathbb{F} \exists x \varphi(x) \rightarrow \forall x \varphi(x)$	Assumption
2.	$\mathbb{T} \exists x \varphi(x)$	$\rightarrow \mathbb{F} 1$
3.	$\mathbb{F} \forall x \varphi(x)$	$\rightarrow \mathbb{F} 1$
4.	$\mathbb{T} \varphi(a)$	$\exists \mathbb{T} 2$
5.	$\mathbb{F} \varphi(a)$	$\forall \mathbb{F} 3$
	$\otimes$	

However,  $\exists x \varphi(x) \rightarrow \forall x \varphi(x)$  is not valid.

## tab.4 Tableaux

explanation

We've said what an assumption is, and we've given the rules of inference. **Tableaux** are inductively generated from these: each **tableau** either is a single branch consisting of one or more assumptions, or it results from a **tableau** by applying one of the rules of inference on a branch.

fol:tab:der:  
sec

**Definition tab.2 (Tableau).** A **tableau** for assumptions  $S_1\varphi_1, \dots, S_n\varphi_n$  (where each  $S_i$  is either  $\mathbb{T}$  or  $\mathbb{F}$ ) is a tree of **signed formulas** satisfying the following conditions:

1. The  $n$  topmost **signed formulas** of the tree are  $S_i\varphi_i$ , one below the other.
2. Every **signed formula** in the tree that is not one of the assumptions results from a correct application of an inference rule to a **signed formula** in the branch above it.

A branch of a **tableau** is *closed* iff it contains both  $\mathbb{T}\varphi$  and  $\mathbb{F}\varphi$ , and *open* otherwise. A **tableau** in which every branch is closed is a *closed tableau* (for its set of assumptions). If a **tableau** is not closed, i.e., if it contains at least one open branch, it is *open*.

**Example tab.3.** Every set of assumptions on its own is a **tableau**, but it will generally not be closed. (Obviously, it is closed only if the assumptions already contain a pair of **signed formulas**  $\mathbb{T}\varphi$  and  $\mathbb{F}\varphi$ .)

From a **tableau** (open or closed) we can obtain a new, larger one by applying one of the rules of inference to a **signed formula**  $\varphi$  in it. The rule will append one or more **signed formulas** to the end of any branch containing the occurrence of  $\varphi$  to which we apply the rule.

For instance, consider the assumption  $\mathbb{T}\varphi \wedge \neg\varphi$ . Here is the (open) **tableau** consisting of just that assumption:

- |    |  |            |
|----|--|------------|
| 1. | $\mathbb{T}\varphi \wedge \neg\varphi$ | Assumption |
|----|--|------------|

We obtain a new **tableau** from it by applying the  $\wedge\mathbb{T}$  rule to the assumption. That rule allows us to add two new lines to the **tableau**,  $\mathbb{T}\varphi$  and  $\mathbb{T}\neg\varphi$ :

1.  $\mathbb{T} \varphi \wedge \neg \varphi$  Assumption
2.  $\mathbb{T} \varphi$   $\wedge \mathbb{T} 1$
3.  $\mathbb{T} \neg \varphi$   $\wedge \mathbb{T} 1$

When we write down **tableaux**, we record the rules we've applied on the right (e.g.,  $\wedge \mathbb{T} 1$  means that the **signed formula** on that line is the result of applying the  $\wedge \mathbb{T}$  rule to the **signed formula** on line 1). This new **tableau** now contains additional **signed formulas**, but to only one ( $\mathbb{T} \neg \varphi$ ) can we apply a rule (in this case, the  $\neg \mathbb{T}$  rule). This results in the closed **tableau**

1.  $\mathbb{T} \varphi \wedge \neg \varphi$  Assumption
  2.  $\mathbb{T} \varphi$   $\wedge \mathbb{T} 1$
  3.  $\mathbb{T} \neg \varphi$   $\wedge \mathbb{T} 1$
  4.  $\mathbb{F} \varphi$   $\neg \mathbb{T} 3$
- $\otimes$

## tab.5 Examples of Tableaux

fol:tab:pro:  
sec

**Example tab.4.** Let's find a closed **tableau** for the **sentence**  $(\varphi \wedge \psi) \rightarrow \varphi$ .

We begin by writing the corresponding assumption at the top of the **tableau**.

1.  $\mathbb{F} (\varphi \wedge \psi) \rightarrow \varphi$  Assumption

There is only one assumption, so only one **signed formula** to which we can apply a rule. (For every **signed formula**, there is always at most one rule that can be applied: it's the rule for the corresponding sign and **main operator** of the **sentence**.) In this case, this means, we must apply  $\rightarrow \mathbb{F}$ .

1.  $\mathbb{F} (\varphi \wedge \psi) \rightarrow \varphi \checkmark$  Assumption
2.  $\mathbb{T} \varphi \wedge \psi$   $\rightarrow \mathbb{F} 1$
3.  $\mathbb{F} \varphi$   $\rightarrow \mathbb{F} 1$

To keep track of which **signed formulas** we have applied their corresponding rules to, we write a checkmark next to the sentence. However, *only* write a checkmark if the rule has been applied to all open branches. Once a **signed formula** has had the corresponding rule applied in every open branch, we will not have to return to it and apply the rule again. In this case, there is only one branch, so the rule only has to be applied once. (Note that checkmarks are only a convenience for constructing tableaux and are not officially part of the syntax of tableaux.)

There is one new **signed formula** to which we can apply a rule: the  $\mathbb{T} \varphi \wedge \psi$  on line 3. Applying the  $\wedge \mathbb{T}$  rule results in:

1.	$\mathbb{F}(\varphi \wedge \psi) \rightarrow \varphi \checkmark$	Assumption
2.	$\mathbb{T}\varphi \wedge \psi \checkmark$	$\rightarrow\mathbb{F} 1$
3.	$\mathbb{F}\varphi$	$\rightarrow\mathbb{F} 1$
4.	$\mathbb{T}\varphi$	$\wedge\mathbb{T} 2$
5.	$\mathbb{T}\psi$	$\wedge\mathbb{T} 2$
	$\otimes$	

Since the branch now contains both  $\mathbb{T}\varphi$  (on line 4) and  $\mathbb{F}\varphi$  (on line 3), the branch is closed. Since it is the only branch, the **tableau** is closed. We have found a closed **tableau** for  $(\varphi \wedge \psi) \rightarrow \varphi$ .

**Example tab.5.** Now let's find a closed **tableau** for  $(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$ .

We begin with the corresponding assumption:

1.	$\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$	Assumption
----	--	------------

The one **signed formula** in this **tableau** has **main operator**  $\rightarrow$  and sign  $\mathbb{F}$ , so we apply the  $\rightarrow\mathbb{F}$  rule to it to obtain:

1.	$\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$	Assumption
2.	$\mathbb{T}\neg\varphi \vee \psi$	$\rightarrow\mathbb{F} 1$
3.	$\mathbb{F}(\varphi \rightarrow \psi)$	$\rightarrow\mathbb{F} 1$

We now have a choice as to whether to apply  $\vee\mathbb{T}$  to line 2 or  $\rightarrow\mathbb{F}$  to line 3. It actually doesn't matter which order we pick, as long as each **signed formula** has its corresponding rule applied in every branch. So let's pick the first one. The  $\vee\mathbb{T}$  rule allows the **tableau** to branch, and the two conclusions of the rule will be the new **signed formulas** added to the two new branches. This results in:

1.	$\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$	Assumption
2.	$\mathbb{T}\neg\varphi \vee \psi \checkmark$	$\rightarrow\mathbb{F} 1$
3.	$\mathbb{F}(\varphi \rightarrow \psi)$	$\rightarrow\mathbb{F} 1$
4.	$\begin{array}{c} \diagup \quad \diagdown \\ \mathbb{T}\neg\varphi \quad \mathbb{T}\psi \end{array}$	$\vee\mathbb{T} 2$

We have not applied the  $\rightarrow\mathbb{F}$  rule to line 3 yet: let's do that now. To save time, we apply it to both branches. Recall that we write a checkmark next to a **signed formula** only if we have applied the corresponding rule in every open branch. So it's a good idea to apply a rule at the end of every branch that contains the **signed formula** the rule applies to. That way we won't have to return to that **signed formula** lower down in the various branches.

1.	$\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$	Assumption	
2.	$\mathbb{T}\neg\varphi \vee \psi \checkmark$	$\rightarrow\mathbb{F}1$	
3.	$\mathbb{F}(\varphi \rightarrow \psi) \checkmark$	$\rightarrow\mathbb{F}1$	
	$\swarrow \quad \searrow$		
4.	$\mathbb{T}\neg\varphi$	$\mathbb{T}\psi$	$\vee\mathbb{T}2$
5.	$\mathbb{T}\varphi$	$\mathbb{T}\varphi$	$\rightarrow\mathbb{F}3$
6.	$\mathbb{F}\psi$	$\mathbb{F}\psi$	$\rightarrow\mathbb{F}3$
	$\otimes$		

The right branch is now closed. On the left branch, we can still apply the  $\neg\mathbb{T}$  rule to line 4. This results in  $\mathbb{F}\varphi$  and closes the left branch:

1.	$\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$	Assumption	
2.	$\mathbb{T}\neg\varphi \vee \psi \checkmark$	$\rightarrow\mathbb{F}1$	
3.	$\mathbb{F}(\varphi \rightarrow \psi) \checkmark$	$\rightarrow\mathbb{F}1$	
	$\swarrow \quad \searrow$		
4.	$\mathbb{T}\neg\varphi$	$\mathbb{T}\psi$	$\vee\mathbb{T}2$
5.	$\mathbb{T}\varphi$	$\mathbb{T}\varphi$	$\rightarrow\mathbb{F}3$
6.	$\mathbb{F}\psi$	$\mathbb{F}\psi$	$\rightarrow\mathbb{F}3$
7.	$\mathbb{F}\varphi$	$\otimes$	$\neg\mathbb{T}4$
	$\otimes$		

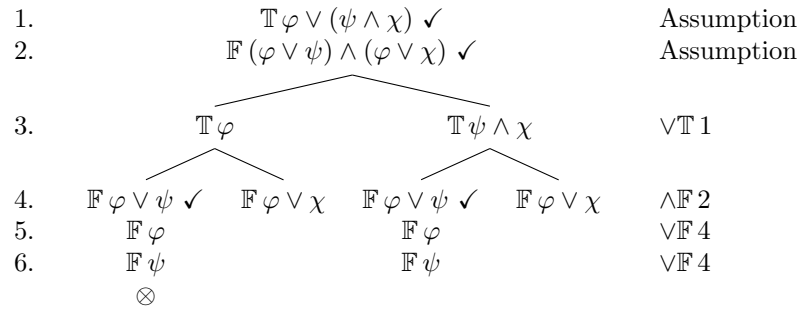
**Example tab.6.** We can give **tableaux** for any number of **signed formulas** as assumptions. Often it is also necessary to apply more than one rule that allows branching; and in general a **tableau** can have any number of branches. For instance, consider a **tableau** for  $\{\mathbb{T}\varphi \vee (\psi \wedge \chi), \mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi)\}$ . We start by applying the  $\vee\mathbb{T}$  to the first assumption:

1.	$\mathbb{T}\varphi \vee (\psi \wedge \chi) \checkmark$	Assumption	
2.	$\mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi)$	Assumption	
	$\swarrow \quad \searrow$		
3.	$\mathbb{T}\varphi$	$\mathbb{T}\psi \wedge \chi$	$\vee\mathbb{T}1$

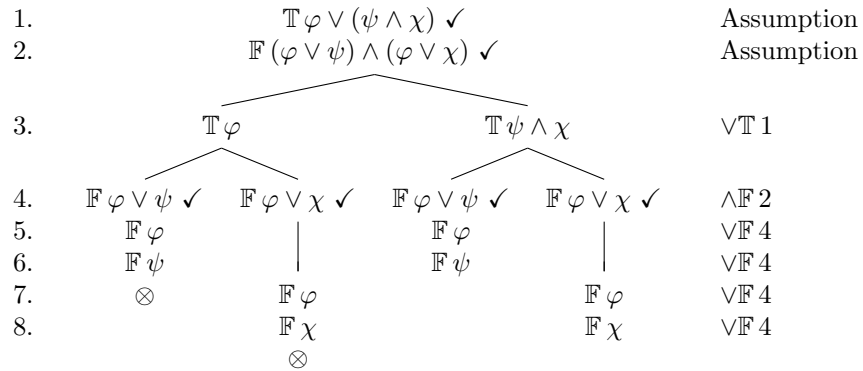
Now we can apply the  $\wedge\mathbb{F}$  rule to line 2. We do this on both branches simultaneously, and can therefore check off line 2:

1.	$\mathbb{T}\varphi \vee (\psi \wedge \chi) \checkmark$	Assumption			
2.	$\mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$	Assumption			
	$\swarrow \quad \searrow$				
3.	$\mathbb{T}\varphi$	$\mathbb{T}\psi \wedge \chi$	$\vee\mathbb{T}1$		
	$\swarrow \quad \searrow \quad \swarrow \quad \searrow$				
4.	$\mathbb{F}\varphi \vee \psi$	$\mathbb{F}\varphi \vee \chi$	$\mathbb{F}\varphi \vee \psi$	$\mathbb{F}\varphi \vee \chi$	$\wedge\mathbb{F}2$

Now we can apply  $\vee\mathbb{F}$  to all the branches containing  $\varphi \vee \psi$ :

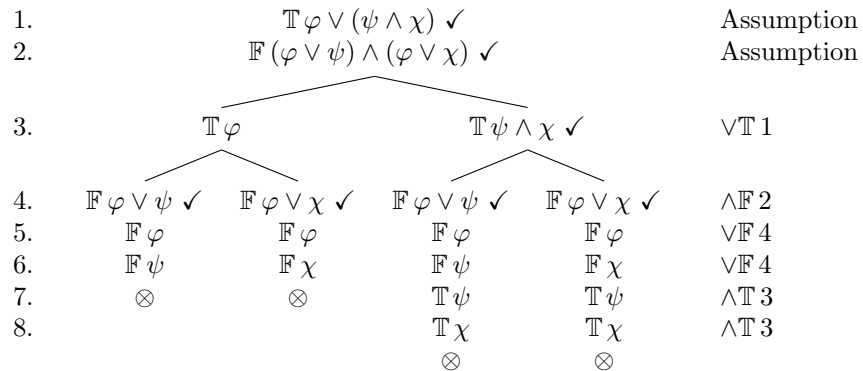


The leftmost branch is now closed. Let's now apply  $\vee\mathbb{F}$  to  $\varphi \vee \chi$ :



Note that we moved the result of applying  $\vee\mathbb{F}$  a second time below for clarity. In this instance it would not have been needed, since the justifications would have been the same.

Two branches remain open, and  $\mathbb{T} \psi \wedge \chi$  on line 3 remains unchecked. We apply  $\wedge\mathbb{T}$  to it to obtain a closed **tableau**:



For comparison, here's a closed **tableau** for the same set of assumptions in which the rules are applied in a different order:



1.	$\mathbb{T} \varphi \vee (\psi \wedge \chi) \checkmark$	Assumption
2.	$\mathbb{F} (\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$	Assumption
$\swarrow$		
3.	$\mathbb{F} \varphi \vee \psi \checkmark$	$\wedge \mathbb{F} 2$
4.	$\mathbb{F} \varphi$	$\vee \mathbb{F} 3$
5.	$\mathbb{F} \psi$	$\vee \mathbb{F} 3$
$\swarrow$		
6.	$\mathbb{T} \varphi$	$\vee \mathbb{T} 1$
7.	$\otimes$	$\wedge \mathbb{T} 3$
8.	$\otimes$	$\wedge \mathbb{T} 3$

**Problem tab.1.** Give closed **tableaux** of the following:

1.  $\mathbb{F} \neg(\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \neg\psi)$
2.  $\mathbb{F} (\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi), \mathbb{T} (\varphi \wedge \psi) \rightarrow \chi$

## tab.6 Tableaux with Quantifiers

fol:tab:prq:  
sec

**Example tab.7.** When dealing with quantifiers, we have to make sure not to violate the eigenvariable condition, and sometimes this requires us to play around with the order of carrying out certain inferences. In general, it helps to try and take care of rules subject to the eigenvariable condition first (they will be higher up in the finished **tableau**).

Let's see how we'd give a **tableau** for the **sentence**  $\exists x \neg\varphi(x) \rightarrow \neg\forall x \varphi(x)$ . Starting as usual, we start by recording the assumption,

1.  $\mathbb{F} \exists x \neg\varphi(x) \rightarrow \neg\forall x \varphi(x)$  Assumption

Since the **main operator** is  $\rightarrow$ , we apply the  $\rightarrow\mathbb{F}$ :

1.  $\mathbb{F} \exists x \neg\varphi(x) \rightarrow \neg\forall x \varphi(x) \checkmark$  Assumption
2.  $\mathbb{F} \exists x \neg\varphi(x)$   $\rightarrow\mathbb{F} 1$
3.  $\mathbb{F} \neg\forall x \varphi(x)$   $\rightarrow\mathbb{F} 1$

The next line to deal with is 2. We use  $\exists\mathbb{T}$ . This requires a new **constant symbol**; since no **constant symbols** yet occur, we can pick any one, say,  $a$ .

1.  $\mathbb{F} \exists x \neg\varphi(x) \rightarrow \neg\forall x \varphi(x) \checkmark$  Assumption
2.  $\mathbb{T} \exists x \neg\varphi(x) \checkmark$   $\rightarrow\mathbb{F} 1$
3.  $\mathbb{F} \neg\forall x \varphi(x)$   $\rightarrow\mathbb{F} 1$
4.  $\mathbb{T} \neg\varphi(a)$   $\exists\mathbb{T} 2$

Now we apply  $\neg\mathbb{F}$  to line 3:

1.	$\mathbb{F} \exists x \neg \varphi(x) \rightarrow \neg \forall x \varphi(x) \checkmark$	Assumption
2.	$\mathbb{T} \exists x \neg \varphi(x) \checkmark$	$\rightarrow \mathbb{F} 1$
3.	$\mathbb{F} \neg \forall x \varphi(x) \checkmark$	$\rightarrow \mathbb{F} 1$
4.	$\mathbb{T} \neg \varphi(a)$	$\exists \mathbb{T} 2$
5.	$\mathbb{T} \forall x \varphi(x)$	$\neg \mathbb{F} 3$

We obtain a closed **tableau** by applying  $\neg \mathbb{T}$  to line 4, followed by  $\forall \mathbb{T}$  to line 5.

1.	$\mathbb{F} \exists x \neg \varphi(x) \rightarrow \neg \forall x \varphi(x) \checkmark$	Assumption
2.	$\mathbb{T} \exists x \neg \varphi(x) \checkmark$	$\rightarrow \mathbb{F} 1$
3.	$\mathbb{F} \neg \forall x \varphi(x) \checkmark$	$\rightarrow \mathbb{F} 1$
4.	$\mathbb{T} \neg \varphi(a)$	$\exists \mathbb{T} 2$
5.	$\mathbb{T} \forall x \varphi(x)$	$\neg \mathbb{F} 3$
6.	$\mathbb{F} \varphi(a)$	$\neg \mathbb{T} 4$
7.	$\mathbb{T} \varphi(a)$	$\forall \mathbb{T} 5$
	$\otimes$	

**Example tab.8.** Let's see how we'd give a **tableau** for the set

$$\mathbb{F} \exists x \chi(x, b), \mathbb{T} \exists x (\varphi(x) \wedge \psi(x)), \mathbb{T} \forall x (\psi(x) \rightarrow \chi(x, b)).$$

Starting as usual, we start with the assumptions:

1.	$\mathbb{F} \exists x \chi(x, b)$	Assumption
2.	$\mathbb{T} \exists x (\varphi(x) \wedge \psi(x))$	Assumption
3.	$\mathbb{T} \forall x (\psi(x) \rightarrow \chi(x, b))$	Assumption

We should always apply a rule with the eigenvariable condition first; in this case that would be  $\exists \mathbb{T}$  to line 2. Since the assumptions contain the **constant symbol**  $b$ , we have to use a different one; let's pick  $a$  again.

1.	$\mathbb{F} \exists x \chi(x, b)$	Assumption
2.	$\mathbb{T} \exists x (\varphi(x) \wedge \psi(x)) \checkmark$	Assumption
3.	$\mathbb{T} \forall x (\psi(x) \rightarrow \chi(x, b))$	Assumption
4.	$\mathbb{T} \varphi(a) \wedge \psi(a)$	$\exists \mathbb{T} 2$

If we now apply  $\exists \mathbb{F}$  to line 1 or  $\forall \mathbb{T}$  to line 3, we have to decide with term  $t$  to substitute for  $x$ . Since there is no eigenvariable condition for these rules, we can pick any term we like. In some cases we may even have to apply the rule several times with different  $ts$ . But as a general rule, it pays to pick one of the terms already occurring in the **tableau**—in this case,  $a$  and  $b$ —and in this case we can guess that  $a$  will be more likely to result in a closed branch.

1.	$\mathbb{F} \exists x \chi(x, b)$	Assumption
2.	$\mathbb{T} \exists x (\varphi(x) \wedge \psi(x)) \checkmark$	Assumption
3.	$\mathbb{T} \forall x (\psi(x) \rightarrow \chi(x, b))$	Assumption
4.	$\mathbb{T} \varphi(a) \wedge \psi(a)$	$\exists \mathbb{T} 2$
5.	$\mathbb{F} \chi(a, b)$	$\exists \mathbb{F} 1$
6.	$\mathbb{T} \psi(a) \rightarrow \chi(a, b)$	$\forall \mathbb{T} 3$

We don't check the **signed formulas** in lines 1 and 3, since we may have to use them again. Now apply  $\wedge\mathbb{T}$  to line 4:

1.	$\mathbb{F} \exists x \chi(x, b)$	Assumption
2.	$\mathbb{T} \exists x (\varphi(x) \wedge \psi(x)) \checkmark$	Assumption
3.	$\mathbb{T} \forall x (\psi(x) \rightarrow \chi(x, b))$	Assumption
4.	$\mathbb{T} \varphi(a) \wedge \psi(a) \checkmark$	$\exists\mathbb{T}$ 2
5.	$\mathbb{F} \chi(a, b)$	$\exists\mathbb{F}$ 1
6.	$\mathbb{T} \psi(a) \rightarrow \chi(a, b)$	$\forall\mathbb{T}$ 1
7.	$\mathbb{T} \varphi(a)$	$\wedge\mathbb{T}$ 4
8.	$\mathbb{T} \psi(a)$	$\wedge\mathbb{T}$ 4

If we now apply  $\rightarrow\mathbb{T}$  to line 5, the **tableau** closes:

1.	$\mathbb{F} \exists x \chi(x, b)$	Assumption
2.	$\mathbb{T} \exists x (\varphi(x) \wedge \psi(x)) \checkmark$	Assumption
3.	$\mathbb{T} \forall x (\psi(x) \rightarrow \chi(x, b))$	Assumption
4.	$\mathbb{T} \varphi(a) \wedge \psi(a) \checkmark$	$\exists\mathbb{T}$ 2
5.	$\mathbb{F} \chi(a, b)$	$\exists\mathbb{F}$ 1
6.	$\mathbb{T} \psi(a) \rightarrow \chi(a, b) \checkmark$	$\forall\mathbb{T}$ 1
7.	$\mathbb{T} \varphi(a)$	$\wedge\mathbb{T}$ 4
8.	$\mathbb{T} \psi(a)$	$\wedge\mathbb{T}$ 4
$\swarrow \quad \searrow$		
9.	$\mathbb{F} \psi(a) \quad \mathbb{T} \chi(a, b)$	$\rightarrow\mathbb{T}$ 6
	$\otimes \quad \otimes$	

**Example tab.9.** We construct a **tableau** for the set

$$\mathbb{T} \forall x \varphi(x), \mathbb{T} \forall x \varphi(x) \rightarrow \exists y \psi(y), \mathbb{T} \neg \exists y \psi(y).$$

Starting as usual, we write down the assumptions:

1.	$\mathbb{T} \forall x \varphi(x)$	Assumption
2.	$\mathbb{T} \forall x \varphi(x) \rightarrow \exists y \psi(y)$	Assumption
3.	$\mathbb{T} \neg \exists y \psi(y)$	Assumption

We begin by applying the  $\neg\mathbb{T}$  rule to line 3. A corollary to the rule “always apply rules with eigenvariable conditions first” is “defer applying quantifier rules without eigenvariable conditions until needed.” Also, defer rules that result in a split.

1.	$\mathbb{T} \forall x \varphi(x)$	Assumption
2.	$\mathbb{T} \forall x \varphi(x) \rightarrow \exists y \psi(y)$	Assumption
3.	$\mathbb{T} \neg \exists y \psi(y) \checkmark$	Assumption
4.	$\mathbb{F} \exists y \psi(y)$	$\neg\mathbb{T}$ 3

The new line 4 requires  $\exists\mathbb{F}$ , a quantifier rule without the eigenvariable condition. So we defer this in favor of using  $\rightarrow\mathbb{T}$  on line 2.

1.	$\mathbb{T}\forall x \varphi(x)$	Assumption
2.	$\mathbb{T}\forall x \varphi(x) \rightarrow \exists y \psi(y) \checkmark$	Assumption
3.	$\mathbb{T}\neg\exists y \psi(y) \checkmark$	Assumption
4.	$\mathbb{F}\exists y \psi(y)$	$\neg\mathbb{T}$ 3
$\swarrow$		
5.	$\mathbb{F}\forall x \varphi(x) \quad \mathbb{T}\exists y \psi(y)$	$\rightarrow\mathbb{T}$ 2

Both new **signed formulas** require rules with eigenvariable conditions, so these should be next:

1.	$\mathbb{T}\forall x \varphi(x)$	Assumption
2.	$\mathbb{T}\forall x \varphi(x) \rightarrow \exists y \psi(y) \checkmark$	Assumption
3.	$\mathbb{T}\neg\exists y \psi(y) \checkmark$	Assumption
4.	$\mathbb{F}\exists y \psi(y) \checkmark$	$\neg\mathbb{T}$ 3
$\swarrow$		
5.	$\mathbb{F}\forall x \varphi(x) \quad \mathbb{T}\exists y \psi(y)$	$\rightarrow\mathbb{T}$ 2
6.	$\mathbb{F}\varphi(b) \quad \mathbb{T}\varphi(c)$	$\forall\mathbb{F}$ 5

To close the branches, we have to use the **signed formulas** on lines 1 and 3. The corresponding rules ( $\forall\mathbb{T}$  and  $\exists\mathbb{F}$ ) don't have eigenvariable conditions, so we are free to pick whichever terms are suitable. In this case, that's  $b$  and  $c$ , respectively.

1.	$\mathbb{T}\forall x \varphi(x)$	Assumption
2.	$\mathbb{T}\forall x \varphi(x) \rightarrow \exists y \psi(y) \checkmark$	Assumption
3.	$\mathbb{T}\neg\exists y \psi(y) \checkmark$	Assumption
4.	$\mathbb{F}\exists y \psi(y) \checkmark$	$\neg\mathbb{T}$ 3
$\swarrow$		
5.	$\mathbb{F}\forall x \varphi(x) \quad \mathbb{T}\exists y \psi(y)$	$\rightarrow\mathbb{T}$ 2
6.	$\mathbb{F}\varphi(b) \quad \mathbb{T}\psi(c)$	$\forall\mathbb{F}$ 5; $\exists\mathbb{T}$ 5
7.	$\mathbb{T}\varphi(b) \quad \mathbb{F}\psi(c)$	$\forall\mathbb{T}$ 1; $\exists\mathbb{F}$ 4

**Problem tab.2.** Give closed **tableaux** of the following:

1.  $\mathbb{F}\exists y \varphi(y) \rightarrow \psi, \mathbb{T}\forall x (\varphi(x) \rightarrow \psi)$
2.  $\mathbb{F}\exists x (\varphi(x) \rightarrow \forall y \varphi(y))$

## tab.7 Proof-Theoretic Notions

fol:tab:ptn:  
sec

This section collects the definitions of the provability relation and consistency for tableaux.

Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the existence of certain closed **tableaux**. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*. explanation

**Definition tab.10** (Theorems). A **sentence**  $\varphi$  is a *theorem* if there is a closed **tableau** for  $\mathbb{F}\varphi$ . We write  $\vdash \varphi$  if  $\varphi$  is a theorem and  $\not\vdash \varphi$  if it is not.

**Definition tab.11** (Derivability). A **sentence**  $\varphi$  is *derivable* from a set of **sentences**  $\Gamma$ ,  $\Gamma \vdash \varphi$ , iff there is a finite set  $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$  and a closed **tableau** for the set

$$\{\mathbb{F}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n, \}$$

If  $\varphi$  is not *derivable* from  $\Gamma$  we write  $\Gamma \not\vdash \varphi$ .

**Definition tab.12** (Consistency). A set of **sentences**  $\Gamma$  is *inconsistent* iff there is a finite set  $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$  and a closed **tableau** for the set

$$\{\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n, \}.$$

If  $\Gamma$  is not inconsistent, we say it is *consistent*.

fol:tab:ptn:  
prop:reflexivity

**Proposition tab.13** (Reflexivity). *If  $\varphi \in \Gamma$ , then  $\Gamma \vdash \varphi$ .*

*Proof.* If  $\varphi \in \Gamma$ ,  $\{\varphi\}$  is a finite subset of  $\Gamma$  and the **tableau**

1.  $\mathbb{F}\varphi$       Assumption
  2.  $\mathbb{T}\varphi$       Assumption
- ⊗

is closed. □

fol:tab:ptn:  
prop:monotony

**Proposition tab.14** (Monotony). *If  $\Gamma \subseteq \Delta$  and  $\Gamma \vdash \varphi$ , then  $\Delta \vdash \varphi$ .*

*Proof.* Any finite subset of  $\Gamma$  is also a finite subset of  $\Delta$ . □

fol:tab:ptn:  
prop:transitivity

**Proposition tab.15** (Transitivity). *If  $\Gamma \vdash \varphi$  and  $\{\varphi\} \cup \Delta \vdash \psi$ , then  $\Gamma \cup \Delta \vdash \psi$ .*

*Proof.* If  $\{\varphi\} \cup \Delta \vdash \psi$ , then there is a finite subset  $\Delta_0 = \{\chi_1, \dots, \chi_n\} \subseteq \Delta$  such that

$$\{\mathbb{F}\psi, \mathbb{T}\varphi, \mathbb{T}\chi_1, \dots, \mathbb{T}\chi_n\}$$

has a closed **tableau**. If  $\Gamma \vdash \varphi$  then there are  $\theta_1, \dots, \theta_m$  such that

$$\{\mathbb{F} \varphi, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m\}$$

has a closed **tableau**.

Now consider the **tableau** with assumptions

$$\mathbb{F} \psi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m.$$

Apply the Cut rule on  $\varphi$ . This generates two branches, one has  $\mathbb{T} \varphi$  in it, the other  $\mathbb{F} \varphi$ . Thus, on the one branch, all of

$$\{\mathbb{F} \psi, \mathbb{T} \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n\}$$

are available. Since there is a closed **tableau** for these assumptions, we can attach it to that branch; every branch through  $\mathbb{T} \varphi_1$  closes. On the other branch, all of

$$\{\mathbb{F} \varphi, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m\}$$

are available, so we can also complete the other side to obtain a closed **tableau**. This shows  $\Gamma \cup \Delta \vdash \psi$ .  $\square$

Note that this means that in particular if  $\Gamma \vdash \varphi$  and  $\varphi \vdash \psi$ , then  $\Gamma \vdash \psi$ . It follows also that if  $\varphi_1, \dots, \varphi_n \vdash \psi$  and  $\Gamma \vdash \varphi_i$  for each  $i$ , then  $\Gamma \vdash \psi$ .

**Proposition tab.16.**  $\Gamma$  is inconsistent iff  $\Gamma \vdash \varphi$  for every **sentence**  $\varphi$ .

*fol:tab:ptn:  
prop:incons*

*Proof.* Exercise.  $\square$

**Problem tab.3.** Prove [Proposition tab.16](#)

**Proposition tab.17** (Compactness).

*fol:tab:ptn:  
prop:proves-compact*

1. If  $\Gamma \vdash \varphi$  then there is a finite subset  $\Gamma_0 \subseteq \Gamma$  such that  $\Gamma_0 \vdash \varphi$ .
2. If every finite subset of  $\Gamma$  is consistent, then  $\Gamma$  is consistent.

*Proof.* 1. If  $\Gamma \vdash \varphi$ , then there is a finite subset  $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$  and a closed **tableau** for

$$\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n$$

This **tableau** also shows  $\Gamma_0 \vdash \varphi$ .

2. If  $\Gamma$  is inconsistent, then for some finite subset  $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$  there is a closed **tableau** for

$$\mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n$$

This closed **tableau** shows that  $\Gamma_0$  is inconsistent.  $\square$

## tab.8 Derivability and Consistency

fol:tab:prv:  
sec

We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:tab:prv:  
prop:provability-contr

**Proposition tab.18.** *If  $\Gamma \vdash \varphi$  and  $\Gamma \cup \{\varphi\}$  is inconsistent, then  $\Gamma$  is inconsistent.*

*Proof.* There are finite  $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$  and  $\Gamma_1 = \{\chi_1, \dots, \chi_m\} \subseteq \Gamma$  such that

$$\begin{aligned} & \{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\} \\ & \{\mathbb{T} \neg \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_m\} \end{aligned}$$

have closed **tableaux**. Using the Cut rule on  $\varphi$  we can combine these into a single closed **tableau** that shows  $\Gamma_0 \cup \Gamma_1$  is inconsistent. Since  $\Gamma_0 \subseteq \Gamma$  and  $\Gamma_1 \subseteq \Gamma$ ,  $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$ , hence  $\Gamma$  is inconsistent.  $\square$

fol:tab:prv:  
prop:prov-incons

**Proposition tab.19.**  *$\Gamma \vdash \varphi$  iff  $\Gamma \cup \{\neg \varphi\}$  is inconsistent.*

*Proof.* First suppose  $\Gamma \vdash \varphi$ , i.e., there is a closed **tableau** for

$$\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}$$

Using the  $\neg\mathbb{T}$  rule, this can be turned into a closed **tableau** for

$$\{\mathbb{T} \neg \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}.$$

On the other hand, if there is a closed **tableau** for the latter, we can turn it into a closed **tableau** of the former by removing every formula that results from  $\neg\mathbb{T}$  applied to the first assumption  $\mathbb{T} \neg \varphi$  as well as that assumption, and adding the assumption  $\mathbb{F} \varphi$ . For if a branch was closed before because it contained the conclusion of  $\neg\mathbb{T}$  applied to  $\mathbb{T} \neg \varphi$ , i.e.,  $\mathbb{F} \varphi$ , the corresponding branch in the new **tableau** is also closed. If a branch in the old tableau was closed because it contained the assumption  $\mathbb{T} \neg \varphi$  as well as  $\mathbb{F} \neg \varphi$  we can turn it into a closed branch by applying  $\neg\mathbb{F}$  to  $\mathbb{F} \neg \varphi$  to obtain  $\mathbb{T} \varphi$ . This closes the branch since we added  $\mathbb{F} \varphi$  as an assumption.  $\square$

**Problem tab.4.** Prove that  $\Gamma \vdash \neg \varphi$  iff  $\Gamma \cup \{\varphi\}$  is inconsistent.

fol:tab:prv:  
prop:explicit-inc

**Proposition tab.20.** *If  $\Gamma \vdash \varphi$  and  $\neg \varphi \in \Gamma$ , then  $\Gamma$  is inconsistent.*

*Proof.* Suppose  $\Gamma \vdash \varphi$  and  $\neg \varphi \in \Gamma$ . Then there are  $\psi_1, \dots, \psi_n \in \Gamma$  such that

$$\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}$$

has a closed tableau. Replace the assumption  $\mathbb{F} \varphi$  by  $\mathbb{T} \neg \varphi$ , and insert the conclusion of  $\neg\mathbb{T}$  applied to  $\mathbb{F} \varphi$  after the assumptions. Any **sentence** in the **tableau** justified by appeal to line 1 in the old **tableau** is now justified by appeal to line  $n + 1$ . So if the old **tableau** was closed, the new one is. It shows that  $\Gamma$  is inconsistent, since all assumptions are in  $\Gamma$ .  $\square$

**Proposition tab.21.** *If  $\Gamma \cup \{\varphi\}$  and  $\Gamma \cup \{\neg\varphi\}$  are both inconsistent, then  $\Gamma$  is inconsistent.* fol:tab:prv:  
prop:provability-exhaustive

*Proof.* If there are  $\psi_1, \dots, \psi_n \in \Gamma$  and  $\chi_1, \dots, \chi_m \in \Gamma$  such that

$$\begin{aligned} & \{\mathbb{T}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\} \\ & \{\mathbb{T}\neg\varphi, \mathbb{T}\chi_1, \dots, \mathbb{T}\chi_m\} \end{aligned}$$

both have closed **tableaux**, we can construct a **tableau** that shows that  $\Gamma$  is inconsistent by using as assumptions  $\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n$  together with  $\mathbb{T}\chi_1, \dots, \mathbb{T}\chi_m$ , followed by an application of the Cut rule, yielding two branches, one starting with  $\mathbb{T}\varphi$ , the other with  $\mathbb{F}\varphi$ . Add on the part below the assumptions of the first **tableau** on the left side. Here, every rule application is still correct, and every branch closes. On the right side, add the part below the assumptions of the second **tableau**, with the results of any applications of  $\neg\mathbb{T}$  to  $\mathbb{T}\neg\varphi$  removed.

For if a branch was closed before because it contained the conclusion of  $\neg\mathbb{T}$  applied to  $\mathbb{T}\neg\varphi$ , i.e.,  $\mathbb{F}\varphi$ , as well as  $\mathbb{F}\varphi$ , the corresponding branch in the new **tableau** is also closed. If a branch in the old tableau was closed because it contained the assumption  $\mathbb{T}\neg\varphi$  as well as  $\mathbb{F}\neg\varphi$  we can turn it into a closed branch by applying  $\neg\mathbb{F}$  to  $\mathbb{F}\neg\varphi$  to obtain  $\mathbb{T}\varphi$ . □

## tab.9 Derivability and the Propositional Connectives

**Proposition tab.22.**

1. Both  $\varphi \wedge \psi \vdash \varphi$  and  $\varphi \wedge \psi \vdash \psi$ .
2.  $\varphi, \psi \vdash \varphi \wedge \psi$ .

*Proof.* 1. Both  $\{\mathbb{F}\varphi, \mathbb{T}\varphi \wedge \psi\}$  and  $\{\mathbb{F}\psi, \mathbb{T}\varphi \wedge \psi\}$  have closed **tableaux**

1.	$\mathbb{F}\varphi$	Assumption
2.	$\mathbb{T}\varphi \wedge \psi$	Assumption
3.	$\mathbb{T}\varphi$	$\wedge\mathbb{T} 2$
4.	$\mathbb{T}\psi$	$\wedge\mathbb{T} 2$
	$\otimes$	

1.	$\mathbb{F}\psi$	Assumption
2.	$\mathbb{T}\varphi \wedge \psi$	Assumption
3.	$\mathbb{T}\varphi$	$\wedge\mathbb{T} 2$
4.	$\mathbb{T}\psi$	$\wedge\mathbb{T} 2$
	$\otimes$	

2. Here is a closed **tableau** for  $\{\mathbb{T}\varphi, \mathbb{T}\psi, \mathbb{F}\varphi \wedge \psi\}$ :



1.	$\mathbb{F} \varphi \wedge \psi$	Assumption				
2.	$\mathbb{T} \varphi$	Assumption				
3.	$\mathbb{T} \psi$	Assumption				
$\swarrow \quad \searrow$						
4.	<table style="display: inline-table; border-collapse: collapse; vertical-align: middle;"> <tr> <td style="padding-right: 10px;"><math>\mathbb{F} \varphi</math></td> <td style="padding-right: 10px;"><math>\mathbb{F} \psi</math></td> </tr> <tr> <td style="text-align: center;"><math>\otimes</math></td> <td style="text-align: center;"><math>\otimes</math></td> </tr> </table>	$\mathbb{F} \varphi$	$\mathbb{F} \psi$	$\otimes$	$\otimes$	$\wedge \mathbb{F} 1$
$\mathbb{F} \varphi$	$\mathbb{F} \psi$					
$\otimes$	$\otimes$					

□

*fol:tab:ppr:* **Proposition tab.23.**  
*prop:provability-lor*

1.  $\varphi \vee \psi, \neg\varphi, \neg\psi$  is inconsistent.
2. Both  $\varphi \vdash \varphi \vee \psi$  and  $\psi \vdash \varphi \vee \psi$ .

*Proof.* 1. We give a closed **tableau** of  $\{\mathbb{T} \varphi \vee \psi, \mathbb{T} \neg\varphi, \mathbb{T} \neg\psi\}$ :

1.	$\mathbb{T} \varphi \vee \psi$	Assumption				
2.	$\mathbb{T} \neg\varphi$	Assumption				
3.	$\mathbb{T} \neg\psi$	Assumption				
4.	$\mathbb{F} \varphi$	$\neg \mathbb{T} 2$				
5.	$\mathbb{F} \psi$	$\neg \mathbb{T} 3$				
$\swarrow \quad \searrow$						
6.	<table style="display: inline-table; border-collapse: collapse; vertical-align: middle;"> <tr> <td style="padding-right: 10px;"><math>\mathbb{T} \varphi</math></td> <td style="padding-right: 10px;"><math>\mathbb{T} \psi</math></td> </tr> <tr> <td style="text-align: center;"><math>\otimes</math></td> <td style="text-align: center;"><math>\otimes</math></td> </tr> </table>	$\mathbb{T} \varphi$	$\mathbb{T} \psi$	$\otimes$	$\otimes$	$\vee \mathbb{T} 1$
$\mathbb{T} \varphi$	$\mathbb{T} \psi$					
$\otimes$	$\otimes$					

2. Both  $\{\mathbb{F} \varphi \vee \psi, \mathbb{T} \varphi\}$  and  $\{\mathbb{F} \varphi \vee \psi, \mathbb{T} \psi\}$  have closed **tableaux**:

1.	$\mathbb{F} \varphi \vee \psi$	Assumption
2.	$\mathbb{T} \varphi$	Assumption
3.	$\mathbb{F} \varphi$	$\vee \mathbb{F} 1$
4.	$\mathbb{F} \psi$	$\vee \mathbb{F} 1$
$\otimes$		

1.	$\mathbb{F} \varphi \vee \psi$	Assumption
2.	$\mathbb{T} \psi$	Assumption
3.	$\mathbb{F} \varphi$	$\vee \mathbb{F} 1$
4.	$\mathbb{F} \psi$	$\vee \mathbb{F} 1$
$\otimes$		

□

*fol:tab:ppr:* **Proposition tab.24.**  
*prop:provability-lif*

1.  $\varphi, \varphi \rightarrow \psi \vdash \psi$ .

*fol:tab:ppr:  
prop:provability-lif-left*

2. Both  $\neg\varphi \vdash \varphi \rightarrow \psi$  and  $\psi \vdash \varphi \rightarrow \psi$ .

*fol:tab:ppr:  
prop:provability-lif-right*

*Proof.* 1.  $\{\mathbb{F}\psi, \mathbb{T}\varphi \rightarrow \psi, \mathbb{T}\varphi\}$  has a closed **tableau**:

1.	$\mathbb{F}\psi$	Assumption				
2.	$\mathbb{T}\varphi \rightarrow \psi$	Assumption				
3.	$\mathbb{T}\varphi$	Assumption				
$\swarrow \quad \searrow$						
4.	<table style="display: inline-table; vertical-align: middle;"> <tr> <td style="padding-right: 10px;"><math>\mathbb{F}\varphi</math></td> <td style="padding-right: 10px;"><math>\mathbb{T}\psi</math></td> </tr> <tr> <td style="text-align: center;"><math>\otimes</math></td> <td style="text-align: center;"><math>\otimes</math></td> </tr> </table>	$\mathbb{F}\varphi$	$\mathbb{T}\psi$	$\otimes$	$\otimes$	$\rightarrow\mathbb{T}2$
$\mathbb{F}\varphi$	$\mathbb{T}\psi$					
$\otimes$	$\otimes$					

2. Both  $s\{\mathbb{F}\varphi \rightarrow \psi, \mathbb{T}\neg\varphi\}$  and  $\{\mathbb{F}\varphi \rightarrow \psi, \mathbb{T}\neg\psi\}$  have closed **tableaux**:

1.	$\mathbb{F}\varphi \rightarrow \psi$	Assumption
2.	$\mathbb{T}\neg\varphi$	Assumption
3.	$\mathbb{T}\varphi$	$\rightarrow\mathbb{F}1$
4.	$\mathbb{F}\psi$	$\rightarrow\mathbb{F}1$
5.	$\mathbb{F}\varphi$	$\neg\mathbb{T}2$
$\otimes$		

1.	$\mathbb{F}\varphi \rightarrow \psi$	Assumption
2.	$\mathbb{T}\neg\psi$	Assumption
3.	$\mathbb{T}\varphi$	$\rightarrow\mathbb{F}1$
4.	$\mathbb{F}\psi$	$\rightarrow\mathbb{F}1$
5.	$\mathbb{F}\psi$	$\neg\mathbb{T}2$
$\otimes$		

□

## tab.10 Derivability and the Quantifiers

*fol:tab:qpr:  
sec*

**Theorem tab.25.** *If  $c$  is a constant not occurring in  $\Gamma$  or  $\varphi(x)$  and  $\Gamma \vdash \varphi(c)$ , then  $\Gamma \vdash \forall x \varphi(x)$ .*

*fol:tab:qpr:  
thm:strong-generalization*

*Proof.* Suppose  $\Gamma \vdash \varphi(c)$ , i.e., there are  $\psi_1, \dots, \psi_n \in \Gamma$  and a closed **tableau** for

$$\{\mathbb{F}\varphi(c), \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}.$$

We have to show that there is also a closed **tableau** for

$$\{\mathbb{F} \forall x \varphi(x), \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}.$$

Take the closed **tableau** and replace the first assumption with  $\mathbb{F} \forall x \varphi(x)$ , and insert  $\mathbb{F} \varphi(c)$  after the assumptions.

$$\begin{array}{c} \mathbb{F} \varphi(c) \\ \mathbb{T} \psi_1 \\ \vdots \\ \mathbb{T} \psi_n \\ \bigwedge \end{array} \quad \begin{array}{c} \mathbb{F} \forall x \varphi(x) \\ \mathbb{T} \psi_1 \\ \vdots \\ \mathbb{T} \psi_n \\ \mathbb{F} \varphi(c) \\ \bigwedge \end{array}$$

The tableau is still closed, since all **sentences** available as assumptions before are still available at the top of the **tableau**. The inserted line is the result of a correct application of  $\forall\mathbb{F}$ , since the **constant symbol**  $c$  does not occur in  $\psi_1, \dots, \psi_n$  of  $\forall x \varphi(x)$ , i.e., it does not occur above the inserted line in the new **tableau**.  $\square$

*fol:tab:qpr:*

*prop:provability-quantifiers*

**Proposition tab.26.**

1.  $\varphi(t) \vdash \exists x \varphi(x)$ .
2.  $\forall x \varphi(x) \vdash \varphi(t)$ .

*Proof.* 1. A closed **tableau** for  $\mathbb{F} \exists x \varphi(x), \mathbb{T} \varphi(t)$  is:

$$\begin{array}{lll} 1. & \mathbb{F} \exists x \varphi(x) & \text{Assumption} \\ 2. & \mathbb{T} \varphi(t) & \text{Assumption} \\ 3. & \mathbb{F} \varphi(t) & \exists\mathbb{F} 1 \\ & \otimes & \end{array}$$

2. A closed **tableau** for  $\mathbb{F} \varphi(t), \mathbb{T} \forall x \varphi(x)$ , is:

$$\begin{array}{lll} 1. & \mathbb{F} \varphi(t) & \text{Assumption} \\ 2. & \mathbb{T} \forall x \varphi(x) & \text{Assumption} \\ 3. & \mathbb{T} \varphi(t) & \forall\mathbb{T} 2 \\ & \otimes & \end{array}$$

$\square$

## tab.11 Soundness

explanation

A **derivation** system, such as tableaux, is *sound* if it cannot **derive** things that do not actually hold. Soundness is thus a kind of guaranteed safety property for **derivation** systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

fol:tab:sou:  
sec

1. every **derivable**  $\varphi$  is valid;
2. if a **sentence** is **derivable** from some others, it is also a consequence of them;
3. if a set of **sentences** is inconsistent, it is unsatisfiable.

These are important properties of a **derivation** system. If any of them do not hold, the **derivation** system is deficient—it would **derive** too much. Consequently, establishing the soundness of a **derivation** system is of the utmost importance.

Because all these proof-theoretic properties are defined via closed **tableaux** of some kind or other, proving (1)–(3) above requires proving something about the semantic properties of closed **tableaux**. We will first define what it means for a **signed formula** to be satisfied in a structure, and then show that if a **tableau** is closed, no structure satisfies all its assumptions. (1)–(3) then follow as corollaries from this result.

**Definition tab.27.** A structure  $\mathfrak{M}$  *satisfies* a **signed formula**  $\mathbb{T}\varphi$  iff  $\mathfrak{M} \models \varphi$ , and it *satisfies*  $\mathbb{F}\varphi$  iff  $\mathfrak{M} \not\models \varphi$ .  $\mathfrak{M}$  *satisfies* a set of **signed formulas**  $\Gamma$  iff it satisfies every  $S\varphi \in \Gamma$ .  $\Gamma$  is *satisfiable* if there is a **structure** that satisfies it, and *unsatisfiable* otherwise.

**Theorem tab.28** (Soundness). *If  $\Gamma$  has a closed **tableau**,  $\Gamma$  is unsatisfiable.*

fol:tab:sou:  
thm:tableau-soundness

*Proof.* Let's call a branch of a **tableau** *satisfiable* iff the set of **signed formulas** on it is satisfiable, and let's call a **tableau** *satisfiable* if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable **tableau** by one of the rules of inference always results in a satisfiable **tableau**. This will prove the theorem: any closed **tableau** results by applying rules of inference to the **tableau** consisting only of assumptions from  $\Gamma$ . So if  $\Gamma$  were satisfiable, any **tableau** for it would be satisfiable. A closed **tableau**, however, is clearly not satisfiable: every branch contains both  $\mathbb{T}\varphi$  and  $\mathbb{F}\varphi$ , and no structure can both satisfy and not satisfy  $\varphi$ .

Suppose we have a satisfiable **tableau**, i.e., a **tableau** with at least one satisfiable branch. Applying a rule of inference either adds **signed formulas** to a branch, or splits a branch in two. If the **tableau** has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended **tableau**, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let  $\Gamma$  be the set of **signed formulas** on that branch, and let  $S\varphi \in \Gamma$  be the **signed formula** to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, i.e.,  $\Gamma$  together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable.

First, we consider the possible inferences with only one premise.

1. The branch is expanded by applying  $\neg\mathbb{T}$  to  $\mathbb{T}\neg\psi \in \Gamma$ . Then the extended branch contains the **signed formulas**  $\Gamma \cup \{\mathbb{F}\psi\}$ . Suppose  $\mathfrak{M} \models \Gamma$ . In particular,  $\mathfrak{M} \models \neg\psi$ . Thus,  $\mathfrak{M} \not\models \psi$ , i.e.,  $\mathfrak{M}$  satisfies  $\mathbb{F}\psi$ .
2. The branch is expanded by applying  $\neg\mathbb{F}$  to  $\mathbb{F}\neg\psi \in \Gamma$ : Exercise.
3. The branch is expanded by applying  $\wedge\mathbb{T}$  to  $\mathbb{T}\psi \wedge \chi \in \Gamma$ , which results in two new **signed formulas** on the branch:  $\mathbb{T}\psi$  and  $\mathbb{T}\chi$ . Suppose  $\mathfrak{M} \models \Gamma$ , in particular  $\mathfrak{M} \models \psi \wedge \chi$ . Then  $\mathfrak{M} \models \psi$  and  $\mathfrak{M} \models \chi$ . This means that  $\mathfrak{M}$  satisfies both  $\mathbb{T}\psi$  and  $\mathbb{T}\chi$ .
4. The branch is expanded by applying  $\vee\mathbb{F}$  to  $\mathbb{T}\psi \vee \chi \in \Gamma$ : Exercise.
5. The branch is expanded by applying  $\rightarrow\mathbb{F}$  to  $\mathbb{T}\psi \rightarrow \chi \in \Gamma$ : This results in two new **signed formulas** on the branch:  $\mathbb{T}\psi$  and  $\mathbb{F}\chi$ . Suppose  $\mathfrak{M} \models \Gamma$ , in particular  $\mathfrak{M} \not\models \psi \rightarrow \chi$ . Then  $\mathfrak{M} \models \psi$  and  $\mathfrak{M} \not\models \chi$ . This means that  $\mathfrak{M}$  satisfies both  $\mathbb{T}\psi$  and  $\mathbb{F}\chi$ .
6. The branch is expanded by applying  $\forall\mathbb{T}$  to  $\mathbb{T}\forall x \psi(x) \in \Gamma$ : This results in a new **signed formula**  $\mathbb{T}\varphi(t)$  on the branch. Suppose  $\mathfrak{M} \models \Gamma$ , in particular,  $\mathfrak{M} \models \forall x \psi(x)$ . By ??,  $\mathfrak{M} \models \varphi(t)$ . Consequently,  $\mathfrak{M}$  satisfies  $\mathbb{T}\varphi(t)$ .
7. The branch is expanded by applying  $\forall\mathbb{F}$  to  $\mathbb{F}\forall x \psi(x) \in \Gamma$ : This results in a new **signed formula**  $\mathbb{F}\varphi(a)$  where  $a$  is a **constant symbol** not occurring in  $\Gamma$ . Since  $\Gamma$  is satisfiable, there is a  $\mathfrak{M}$  such that  $\mathfrak{M} \models \Gamma$ , in particular  $\mathfrak{M} \not\models \forall x \psi(x)$ . We have to show that  $\Gamma \cup \{\mathbb{F}\varphi(a)\}$  is satisfiable. To do this, we define a suitable  $\mathfrak{M}'$  as follows.  
 By ??,  $\mathfrak{M} \not\models \forall x \psi(x)$  iff for some  $s$ ,  $\mathfrak{M}, s \not\models \psi(x)$ . Now let  $\mathfrak{M}'$  be just like  $\mathfrak{M}$ , except  $a^{\mathfrak{M}'} = s(x)$ . By ??, for any  $\mathbb{T}\chi \in \Gamma$ ,  $\mathfrak{M}' \models \chi$ , and for any  $\mathbb{F}\chi \in \Gamma$ ,  $\mathfrak{M}' \not\models \chi$ , since  $a$  does not occur in  $\Gamma$ .  
 By ??,  $\mathfrak{M}', s \not\models \varphi(x)$ . By ??,  $\mathfrak{M}', s \not\models \varphi(a)$ . Since  $\varphi(a)$  is a **sentence**, by ??,  $\mathfrak{M}' \not\models \varphi(a)$ , i.e.,  $\mathfrak{M}'$  satisfies  $\mathbb{F}\varphi(a)$ .
8. The branch is expanded by applying  $\exists\mathbb{T}$  to  $\mathbb{T}\exists x \psi(x) \in \Gamma$ : Exercise.
9. The branch is expanded by applying  $\exists\mathbb{F}$  to  $\mathbb{F}\exists x \psi(x) \in \Gamma$ : Exercise.

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying  $\wedge\mathbb{F}$  to  $\mathbb{F}\psi \wedge \chi \in \Gamma$ , which results in two branches, a left one continuing through  $\mathbb{F}\psi$  and a right one through  $\mathbb{F}\chi$ . Suppose  $\mathfrak{M} \models \Gamma$ , in particular  $\mathfrak{M} \not\models \psi \wedge \chi$ . Then  $\mathfrak{M} \not\models \psi$  or  $\mathfrak{M} \not\models \chi$ .

In the former case,  $\mathfrak{M}$  satisfies  $\mathbb{F}\psi$ , i.e.,  $\mathfrak{M}$  satisfies the formulas on the left branch. In the latter,  $\mathfrak{M}$  satisfies  $\mathbb{F}\chi$ , i.e.,  $\mathfrak{M}$  satisfies the formulas on the right branch.

2. The branch is expanded by applying  $\vee\mathbb{T}$  to  $\mathbb{T}\psi \vee \chi \in \Gamma$ : Exercise.
3. The branch is expanded by applying  $\rightarrow\mathbb{T}$  to  $\mathbb{T}\psi \rightarrow \chi \in \Gamma$ : Exercise.
4. The branch is expanded by Cut: This results in two branches, one containing  $\mathbb{T}\psi$ , the other containing  $\mathbb{F}\psi$ . Since  $\mathfrak{M} \models \Gamma$  and either  $\mathfrak{M} \models \psi$  or  $\mathfrak{M} \not\models \psi$ ,  $\mathfrak{M}$  satisfies either the left or the right branch.

□

**Problem tab.5.** Complete the proof of [Theorem tab.28](#).

**Corollary tab.29.** *If  $\Gamma \vdash \varphi$  then  $\varphi$  is valid.*

*fol:tab:sou:  
cor:weak-soundness*

**Corollary tab.30.** *If  $\Gamma \vdash \varphi$  then  $\Gamma \models \varphi$ .*

*fol:tab:sou:  
cor:entailment-soundness*

*Proof.* If  $\Gamma \vdash \varphi$  then for some  $\psi_1, \dots, \psi_n \in \Gamma$ ,  $\{\mathbb{F}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$  has a closed [tableau](#). By [Theorem tab.28](#), every [structure](#)  $\mathfrak{M}$  either makes some  $\psi_i$  false or makes  $\varphi$  true. Hence, if  $\mathfrak{M} \models \Gamma$  then also  $\mathfrak{M} \models \varphi$ . □

**Corollary tab.31.** *If  $\Gamma$  is satisfiable, then it is consistent.*

*fol:tab:sou:  
cor:consistency-soundness*

*Proof.* We prove the contrapositive. Suppose that  $\Gamma$  is not consistent. Then there are  $\psi_1, \dots, \psi_n \in \Gamma$  and a closed [tableau](#) for  $\{\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$ . By [Theorem tab.28](#), there is no  $\mathfrak{M}$  such that  $\mathfrak{M} \models \psi_i$  for all  $i = 1, \dots, n$ . But then  $\Gamma$  is not satisfiable. □

## tab.12 Tableaux with Identity predicate

[Tableaux](#) with [identity predicate](#) require additional inference rules. The rules for  $=$  are ( $t, t_1$ , and  $t_2$  are closed terms):

*fol:tab:ide:  
sec*

$$\boxed{\begin{array}{ccc} \frac{}{\mathbb{T}t = t} = & \frac{\mathbb{T}t_1 = t_2}{\mathbb{T}\varphi(t_1)} = \mathbb{T} & \frac{\mathbb{T}t_1 = t_2}{\mathbb{F}\varphi(t_1)} = \mathbb{T} \end{array}}$$

Note that in contrast to all the other rules,  $=\mathbb{T}$  and  $=\mathbb{F}$  require that *two* signed [formulas](#) already appear on the branch, namely both  $\mathbb{T}t_1 = t_2$  and  $S\varphi(t_1)$ .

**Example tab.32.** If  $s$  and  $t$  are closed terms, then  $s = t, \varphi(s) \vdash \varphi(t)$ :

1.  $\mathbb{F} \varphi(t)$  Assumption
  2.  $\mathbb{T} s = t$  Assumption
  3.  $\mathbb{T} \varphi(s)$  Assumption
  4.  $\mathbb{T} \varphi(t)$   $=\mathbb{T} 2, 3$
- ⊗

This may be familiar as the principle of substitutability of identicals, or Leibniz' Law.

Tableaux prove that  $=$  is symmetric:

1.  $\mathbb{F} t = s$  Assumption
  2.  $\mathbb{T} s = t$  Assumption
  3.  $\mathbb{T} s = s$   $=$
  4.  $\mathbb{T} t = s$   $=\mathbb{T} 2, 3$
- ⊗

Here, line 2 is the first prerequisite formula  $\mathbb{T} s = t$  of  $=\mathbb{T}$ , and line 3 the second one,  $\mathbb{T} \varphi(s)$ —think of  $\varphi(x)$  as  $x = s$ , then  $\varphi(s)$  is  $s = s$  and  $\varphi(t)$  is  $t = s$ .

They also prove that  $=$  is transitive:

1.  $\mathbb{F} t_1 = t_3$  Assumption
  2.  $\mathbb{T} t_1 = t_2$  Assumption
  3.  $\mathbb{T} t_2 = t_3$  Assumption
  4.  $\mathbb{T} t_1 = t_3$   $=\mathbb{T} 3, 2$
- ⊗

In this tableau, the first prerequisite formula of  $=\mathbb{T}$  is line 3,  $\mathbb{T} t_2 = t_3$ . The second one,  $\mathbb{T} \varphi(t_2)$  is line 2. Think of  $\varphi(x)$  as  $t_1 = x$ ; that makes  $\varphi(t_2)$  into  $t_1 = t_2$  and  $\varphi(t_3)$  into  $t_1 = t_3$ .

**Problem tab.6.** Give closed tableaux for the following:

1.  $\mathbb{F} \forall x \forall y ((x = y \wedge \varphi(x)) \rightarrow \varphi(y))$
2.  $\mathbb{F} \exists x (\varphi(x) \wedge \forall y (\varphi(y) \rightarrow y = x)),$   
 $\mathbb{T} \exists x \varphi(x) \wedge \forall y \forall z ((\varphi(y) \wedge \varphi(z)) \rightarrow y = z)$

### tab.13 Soundness with Identity predicate

fol:tab:sid:  
sec

**Proposition tab.33.** *Tableaux with rules for identity are sound: no closed tableau is satisfiable.*

*Proof.* We just have to show as before that if a tableau has a satisfiable branch, the branch resulting from applying one of the rules for  $=$  to it is also satisfiable. Let  $\Gamma$  be the set of signed formulas on the branch, and let  $\mathfrak{M}$  be a structure satisfying  $\Gamma$ .

Suppose the branch is expanded using  $=$ , i.e., by adding the signed formula  $\mathbb{T}t = t$ . Trivially,  $\mathfrak{M} \models t = t$ , so  $\mathfrak{M}$  also satisfies  $\Gamma \cup \{\mathbb{T}t = t\}$ .

If the branch is expanded using  $=\mathbb{T}$ , we add a signed formula  $\mathbb{T}\varphi(t_2)$ , but  $\Gamma$  contains both  $\mathbb{T}t_1 = t_2$  and  $\mathbb{T}\varphi(t_1)$ . Thus we have  $\mathfrak{M} \models t_1 = t_2$  and  $\mathfrak{M} \models \varphi(t_1)$ . Let  $s$  be a variable assignment with  $s(x) = \text{Val}^{\mathfrak{M}}(t_1)$ . By ??,  $\mathfrak{M}, s \models \varphi(t_1)$ . Since  $s \sim_x s$ , by ??,  $\mathfrak{M}, s \models \varphi(x)$ . since  $\mathfrak{M} \models t_1 = t_2$ , we have  $\text{Val}^{\mathfrak{M}}(t_1) = \text{Val}^{\mathfrak{M}}(t_2)$ , and hence  $s(x) = \text{Val}^{\mathfrak{M}}(t_2)$ . By applying ?? again, we also have  $\mathfrak{M}, s \models \varphi(t_2)$ . By ??,  $\mathfrak{M} \models \varphi(t_2)$ . The case of  $=\mathbb{F}$  is treated similarly.  $\square$

## Photo Credits



# Bibliography