Chapter udf

Tableaux

> This chapter presents a signed analytic tableaux system.
>
> To include or exclude material relevant to natural deduction as a proof system, use the "prfTab" tag.

tab.1 Rules and Tableaux

fol:tab:rul:sec A tableau is a systematic survey of the possible ways a sentence can be true or false in a structure. The building blocks of a tableau are signed formulas: sentences plus a truth value "sign," either $\mathbb{T}$ or $\mathbb{F}$. These signed formulas are arranged in a (downward growing) tree.

Definition tab.1. A *signed formula* is a pair consisting of a truth value and a sentence, i.e., either:

$$\mathbb{T}\,\varphi \text{ or } \mathbb{F}\,\varphi.$$

Intuitively, we might read $\mathbb{T}\,\varphi$ as "φ might be true" and $\mathbb{F}\,\varphi$ as "φ might be false" (in some structure).

Each signed formula in the tree is either an *assumption* (which are listed at the very top of the tree), or it is obtained from a signed formula above it by one of a number of rules of inference. There are two rules for each possible main operator of the preceding formula, one for the case where the sign is $\mathbb{T}$, and one for the case where the sign is $\mathbb{F}$. Some rules allow the tree to branch, and some only add signed formulas to the branch. A rule may be (and often must be) applied not to the immediately preceding signed formula, but to any signed formula in the branch from the root to the place the rule is applied.

A branch is *closed* when it contains both $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$. A closed tableau is one where every branch is closed. Under the intuitive interpretation, any branch describes a joint possibility, but $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$ are not jointly possible. In other words, if a branch is closed, the possibility it describes has been ruled out. In particular, that means that a closed tableau rules out all possibilities of simultaneously making every assumption of the form $\mathbb{T}\,\varphi$ true and every assumption of the form $\mathbb{F}\,\varphi$ false.

A closed tableau *for* φ is a closed tableau with root $\mathbb{F}\,\varphi$. If such a closed tableau exists, all possibilities for φ being false have been ruled out; i.e., φ must be true in every structure.

tab.2 Propositional Rules

Rules for $\neg$

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$$\frac{\mathbb{T}\,\neg\varphi}{\mathbb{F}\,\varphi}\,\neg\mathbb{T} \qquad\qquad \frac{\mathbb{F}\,\neg\varphi}{\mathbb{T}\,\varphi}\,\neg\mathbb{F}$$

Rules for $\wedge$

$$\cfrac{\mathbb{T}\,\varphi\wedge\psi}{\begin{array}{c}\mathbb{T}\,\varphi\\ \mathbb{T}\,\psi\end{array}}\,\wedge\mathbb{T} \qquad\qquad \frac{\mathbb{F}\,\varphi\wedge\psi}{\mathbb{F}\,\varphi \quad \mid \quad \mathbb{F}\,\psi}\,\wedge\mathbb{F}$$

Rules for $\vee$

$$\frac{\mathbb{T}\,\varphi\vee\psi}{\mathbb{T}\,\varphi \quad \mid \quad \mathbb{T}\,\psi}\,\vee\mathbb{T} \qquad\qquad \cfrac{\mathbb{F}\,\varphi\vee\psi}{\begin{array}{c}\mathbb{F}\,\varphi\\ \mathbb{F}\,\psi\end{array}}\,\vee\mathbb{F}$$

Rules for $\rightarrow$

$$\frac{\mathbb{T}\,\varphi\rightarrow\psi}{\mathbb{F}\,\varphi \quad \mid \quad \mathbb{T}\,\psi}\,\rightarrow\mathbb{T} \qquad\qquad \cfrac{\mathbb{F}\,\varphi\rightarrow\psi}{\begin{array}{c}\mathbb{T}\,\varphi\\ \mathbb{F}\,\psi\end{array}}\,\rightarrow\mathbb{F}$$

The Cut Rule

$$\frac{}{\mathbb{T}\,\varphi \quad \mid \quad \mathbb{F}\,\varphi}\,\text{Cut}$$

The Cut rule is not applied "to" a previous signed formula; rather, it allows every branch in a tableau to be split in two, one branch containing $\mathbb{T}\,\varphi$, the other $\mathbb{F}\,\varphi$. It is not necessary—any set of signed formulas with a closed tableau has one not using Cut—but it allows us to combine tableaux in a convenient way.

tab.3 Quantifier Rules

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Rules for $\forall$

$$\cfrac{\mathbb{T}\,\forall x\,\varphi(x)}{\mathbb{T}\,\varphi(t)}\;\forall\mathbb{T} \qquad\qquad \cfrac{\mathbb{F}\,\forall x\,\varphi(x)}{\mathbb{F}\,\varphi(a)}\;\forall\mathbb{F}$$

In $\forall\mathbb{T}$, t is a closed term (i.e., one without variables). In $\forall\mathbb{F}$, a is a constant symbol which must not occur anywhere in the branch above $\forall\mathbb{F}$ rule. We call a the *eigenvariable* of the $\forall\mathbb{F}$ inference.[1]

Rules for $\exists$

$$\cfrac{\mathbb{T}\,\exists x\,\varphi(x)}{\mathbb{T}\,\varphi(a)}\;\exists\mathbb{T} \qquad\qquad \cfrac{\mathbb{F}\,\exists x\,\varphi(x)}{\mathbb{F}\,\varphi(t)}\;\exists\mathbb{F}$$

Again, t is a closed term, and a is a constant symbol which does not occur in the branch above the $\exists\mathbb{T}$ rule. We call a the *eigenvariable* of the $\exists\mathbb{T}$ inference.

The condition that an eigenvariable not occur in the branch above the $\forall\mathbb{F}$ or $\exists\mathbb{T}$ inference is called the *eigenvariable condition*.

Recall the convention that when φ is a formula with the variable x free, we indicate this by writing $\varphi(x)$. In the same context, $\varphi(t)$ then is short for $\varphi[t/x]$. So we could also write the $\exists\mathbb{F}$ rule as: explanation

$$\cfrac{\mathbb{F}\,\exists x\,\varphi}{\mathbb{F}\,\varphi[t/x]}\;\exists\mathbb{F}$$

Note that t may already occur in φ, e.g., φ might be $P(t,x)$. Thus, inferring $\mathbb{F}\,P(t,t)$ from $\mathbb{F}\,\exists x\,P(t,x)$ is a correct application of $\exists\mathbb{F}$. However, the eigenvariable conditions in $\forall\mathbb{F}$ and $\exists\mathbb{T}$ require that the constant symbol a does not occur in φ. So, you cannot correctly infer $\mathbb{F}\,P(a,a)$ from $\mathbb{F}\,\forall x\,P(a,x)$ using $\forall\mathbb{F}$.

In $\forall\mathbb{T}$ and $\exists\mathbb{F}$ there are no restrictions on the term t. On the other hand, in the $\exists\mathbb{T}$ and $\forall\mathbb{F}$ rules, the eigenvariable condition requires that the constant symbol a does not occur anywhere in the branches above the respective inference. explanation

[1] We use the term "eigenvariable" even though a in the above rule is a constant symbol. This has historical reasons.

It is necessary to ensure that the system is sound. Without this condition, the following would be a closed tableau for $\exists x\, \varphi(x) \to \forall x\, \varphi(x)$:

1.	$\mathbb{F}\, \exists x\, \varphi(x) \to \forall x\, \varphi(x)$	Assumption
2.	$\mathbb{T}\, \exists x\, \varphi(x)$	$\to\mathbb{F}\, 1$
3.	$\mathbb{F}\, \forall x\, \varphi(x)$	$\to\mathbb{F}\, 1$
4.	$\mathbb{T}\, \varphi(a)$	$\exists\mathbb{T}\, 2$
5.	$\mathbb{F}\, \varphi(a)$	$\forall\mathbb{F}\, 3$
	$\otimes$	

However, $\exists x\, \varphi(x) \to \forall x\, \varphi(x)$ is not valid.

tab.4 Tableaux

explanation We've said what an assumption is, and we've given the rules of inference. Tableaux are inductively generated from these: each tableau either is a single branch consisting of one or more assumptions, or it results from a tableau by applying one of the rules of inference on a branch. fol:tab:der:sec

Definition tab.2 (Tableau). A tableau for assumptions $S_1\varphi_1, \dots, S_n\varphi_n$ (where each S_i is either $\mathbb{T}$ or $\mathbb{F}$) is a finite tree of signed formulas satisfying the following conditions:

1. The n topmost signed formulas of the tree are $S_i\varphi_i$, one below the other.
2. Every signed formula in the tree that is not one of the assumptions results from a correct application of an inference rule to a signed formula in the branch above it.

A branch of a tableau is *closed* iff it contains both $\mathbb{T}\, \varphi$ and $\mathbb{F}\, \varphi$, and *open* otherwise. A tableau in which every branch is closed is a *closed tableau* (for its set of assumptions). If a tableau is not closed, i.e., if it contains at least one open branch, it is *open*.

Example tab.3. Every set of assumptions on its own is a tableau, but it will generally not be closed. (Obviously, it is closed only if the assumptions already contain a pair of signed formulas $\mathbb{T}\, \varphi$ and $\mathbb{F}\, \varphi$.)

From a tableau (open or closed) we can obtain a new, larger one by applying one of the rules of inference to a signed formula φ in it. The rule will append one or more signed formulas to the end of any branch containing the occurrence of φ to which we apply the rule.

For instance, consider the assumption $\mathbb{T}\, \varphi \land \neg\varphi$. Here is the (open) tableau consisting of just that assumption:

1.	$\mathbb{T}\, \varphi \land \neg\varphi$	Assumption

We obtain a new tableau from it by applying the $\land\mathbb{T}$ rule to the assumption. That rule allows us to add two new lines to the tableau, $\mathbb{T}\, \varphi$ and $\mathbb{T}\, \neg\varphi$:

1.	$\mathbb{T}\,\varphi \land \neg\varphi$	Assumption
2.	$\mathbb{T}\,\varphi$	$\land\mathbb{T}\,1$
3.	$\mathbb{T}\,\neg\varphi$	$\land\mathbb{T}\,1$

When we write down tableaux, we record the rules we've applied on the right (e.g., $\land\mathbb{T}1$ means that the signed formula on that line is the result of applying the $\land\mathbb{T}$ rule to the signed formula on line 1). This new tableau now contains additional signed formulas, but to only one ($\mathbb{T}\,\neg\varphi$) can we apply a rule (in this case, the $\neg\mathbb{T}$ rule). This results in the closed tableau

1.	$\mathbb{T}\,\varphi \land \neg\varphi$	Assumption
2.	$\mathbb{T}\,\varphi$	$\land\mathbb{T}\,1$
3.	$\mathbb{T}\,\neg\varphi$	$\land\mathbb{T}\,1$
4.	$\mathbb{F}\,\varphi$	$\neg\mathbb{T}\,3$
	$\otimes$	

tab.5 Examples of Tableaux

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Example tab.4. Let's find a closed tableau for the sentence $(\varphi \land \psi) \to \varphi$.

We begin by writing the corresponding assumption at the top of the tableau.

1.	$\mathbb{F}\,(\varphi \land \psi) \to \varphi$	Assumption

There is only one assumption, so only one signed formula to which we can apply a rule. (For every signed formula, there is always at most one rule that can be applied: it's the rule for the corresponding sign and main operator of the sentence.) In this case, this means, we must apply $\to\mathbb{F}$.

1.	$\mathbb{F}\,(\varphi \land \psi) \to \varphi$ ✓	Assumption
2.	$\mathbb{T}\,\varphi \land \psi$	$\to\mathbb{F}\,1$
3.	$\mathbb{F}\,\varphi$	$\to\mathbb{F}\,1$

To keep track of which signed formulas we have applied their corresponding rules to, we write a checkmark next to the sentence. However, *only* write a checkmark if the rule has been applied to all open branches. Once a signed formula has had the corresponding rule applied in every open branch, we will not have to return to it and apply the rule again. In this case, there is only one branch, so the rule only has to be applied once. (Note that checkmarks are only a convenience for constructing tableaux and are not officially part of the syntax of tableaux.)

There is one new signed formula to which we can apply a rule: the $\mathbb{T}\,\varphi \land \psi$ on line 2. Applying the $\land\mathbb{T}$ rule results in:

$$
\begin{array}{lcl}
1. & \mathbb{F}\,(\varphi \land \psi) \to \varphi \checkmark & \text{Assumption} \\
2. & \mathbb{T}\,\varphi \land \psi \checkmark & {\to}\mathbb{F}\,1 \\
3. & \mathbb{F}\,\varphi & {\to}\mathbb{F}\,1 \\
4. & \mathbb{T}\,\varphi & {\land}\mathbb{T}\,2 \\
5. & \mathbb{T}\,\psi & {\land}\mathbb{T}\,2 \\
 & \otimes &
\end{array}
$$

Since the branch now contains both $\mathbb{T}\,\varphi$ (on line 4) and $\mathbb{F}\,\varphi$ (on line 3), the branch is closed. Since it is the only branch, the tableau is closed. We have found a closed tableau for $(\varphi \land \psi) \to \varphi$.

Example tab.5. Now let's find a closed tableau for $(\neg\varphi \lor \psi) \to (\varphi \to \psi)$.

We begin with the corresponding assumption:

$$
\begin{array}{lcl}
1. & \mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi) & \text{Assumption}
\end{array}
$$

The one signed formula in this tableau has main operator $\to$ and sign $\mathbb{F}$, so we apply the ${\to}\mathbb{F}$ rule to it to obtain:

$$
\begin{array}{lcl}
1. & \mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi) \checkmark & \text{Assumption} \\
2. & \mathbb{T}\,\neg\varphi \lor \psi & {\to}\mathbb{F}\,1 \\
3. & \mathbb{F}\,(\varphi \to \psi) & {\to}\mathbb{F}\,1
\end{array}
$$

We now have a choice as to whether to apply ${\lor}\mathbb{T}$ to line 2 or ${\to}\mathbb{F}$ to line 3. It actually doesn't matter which order we pick, as long as each signed formula has its corresponding rule applied in every branch. So let's pick the first one. The ${\lor}\mathbb{T}$ rule allows the tableau to branch, and the two conclusions of the rule will be the new signed formulas added to the two new branches. This results in:

$$
\begin{array}{lcl}
1. & \mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi) \checkmark & \text{Assumption} \\
2. & \mathbb{T}\,\neg\varphi \lor \psi \checkmark & {\to}\mathbb{F}\,1 \\
3. & \mathbb{F}\,(\varphi \to \psi) & {\to}\mathbb{F}\,1 \\
 & \diagup \quad \diagdown & \\
4. & \mathbb{T}\,\neg\varphi \qquad \mathbb{T}\,\psi & {\lor}\mathbb{T}\,2
\end{array}
$$

We have not applied the ${\to}\mathbb{F}$ rule to line 3 yet: let's do that now. To save time, we apply it to both branches. Recall that we write a checkmark next to a signed formula only if we have applied the corresponding rule in every open branch. So it's a good idea to apply a rule at the end of every branch that contains the signed formula the rule applies to. That way we won't have to return to that signed formula lower down in the various branches.

1.	$\mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi)$ ✓		Assumption
2.	$\mathbb{T}\,\neg\varphi \lor \psi$ ✓		$\to\mathbb{F}\,1$
3.	$\mathbb{F}\,(\varphi \to \psi)$ ✓		$\to\mathbb{F}\,1$
4.	$\mathbb{T}\,\neg\varphi$	$\mathbb{T}\,\psi$	$\lor\mathbb{T}\,2$
5.	$\mathbb{T}\,\varphi$	$\mathbb{T}\,\varphi$	$\to\mathbb{F}\,3$
6.	$\mathbb{F}\,\psi$	$\mathbb{F}\,\psi$	$\to\mathbb{F}\,3$
		$\otimes$	

The right branch is now closed. On the left branch, we can still apply the $\neg\mathbb{T}$ rule to line 4. This results in $\mathbb{F}\,\varphi$ and closes the left branch:

1.	$\mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi)$ ✓		Assumption
2.	$\mathbb{T}\,\neg\varphi \lor \psi$ ✓		$\to\mathbb{F}\,1$
3.	$\mathbb{F}\,(\varphi \to \psi)$ ✓		$\to\mathbb{F}\,1$
4.	$\mathbb{T}\,\neg\varphi$	$\mathbb{T}\,\psi$	$\lor\mathbb{T}\,2$
5.	$\mathbb{T}\,\varphi$	$\mathbb{T}\,\varphi$	$\to\mathbb{F}\,3$
6.	$\mathbb{F}\,\psi$	$\mathbb{F}\,\psi$	$\to\mathbb{F}\,3$
7.	$\mathbb{F}\,\varphi$	$\otimes$	$\neg\mathbb{T}\,4$
	$\otimes$		

Example tab.6. We can give tableaux for any number of signed formulas as assumptions. Often it is also necessary to apply more than one rule that allows branching; and in general a tableau can have any number of branches. For instance, consider a tableau for $\{\mathbb{T}\,\varphi \lor (\psi \land \chi), \mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)\}$. We start by applying the $\lor\mathbb{T}$ to the first assumption:

1.	$\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓		Assumption
2.	$\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$		Assumption
3.	$\mathbb{T}\,\varphi$	$\mathbb{T}\,\psi \land \chi$	$\lor\mathbb{T}\,1$

Now we can apply the $\land\mathbb{F}$ rule to line 2. We do this on both branches simultaneously, and can therefore check off line 2:

1.	$\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓				Assumption
2.	$\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓				Assumption
3.	$\mathbb{T}\,\varphi$		$\mathbb{T}\,\psi \land \chi$		$\lor\mathbb{T}\,1$
4.	$\mathbb{F}\,\varphi \lor \psi$	$\mathbb{F}\,\varphi \lor \chi$	$\mathbb{F}\,\varphi \lor \psi$	$\mathbb{F}\,\varphi \lor \chi$	$\land\mathbb{F}\,2$

Now we can apply $\lor\mathbb{F}$ to all the branches containing $\varphi \lor \psi$:

1.	$\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓				Assumption
2.	$\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓				Assumption
3.	$\mathbb{T}\,\varphi$		$\mathbb{T}\,\psi \land \chi$		$\lor\mathbb{T}\,1$
4.	$\mathbb{F}\,\varphi \lor \psi$ ✓	$\mathbb{F}\,\varphi \lor \chi$	$\mathbb{F}\,\varphi \lor \psi$ ✓	$\mathbb{F}\,\varphi \lor \chi$	$\land\mathbb{F}\,2$
5.	$\mathbb{F}\,\varphi$		$\mathbb{F}\,\varphi$		$\lor\mathbb{F}\,4$
6.	$\mathbb{F}\,\psi$		$\mathbb{F}\,\psi$		$\lor\mathbb{F}\,4$
	$\otimes$				

The leftmost branch is now closed. Let's now apply $\lor\mathbb{F}$ to $\varphi \lor \chi$:

1.	$\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓				Assumption
2.	$\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓				Assumption
3.	$\mathbb{T}\,\varphi$		$\mathbb{T}\,\psi \land \chi$		$\lor\mathbb{T}\,1$
4.	$\mathbb{F}\,\varphi \lor \psi$ ✓	$\mathbb{F}\,\varphi \lor \chi$ ✓	$\mathbb{F}\,\varphi \lor \psi$ ✓	$\mathbb{F}\,\varphi \lor \chi$ ✓	$\land\mathbb{F}\,2$
5.	$\mathbb{F}\,\varphi$		$\mathbb{F}\,\varphi$		$\lor\mathbb{F}\,4$
6.	$\mathbb{F}\,\psi$		$\mathbb{F}\,\psi$		$\lor\mathbb{F}\,4$
7.	$\otimes$	$\mathbb{F}\,\varphi$		$\mathbb{F}\,\varphi$	$\lor\mathbb{F}\,4$
8.		$\mathbb{F}\,\chi$		$\mathbb{F}\,\chi$	$\lor\mathbb{F}\,4$
		$\otimes$			

Note that we moved the result of applying $\lor\mathbb{F}$ a second time below for clarity. In this instance it would not have been needed, since the justifications would have been the same.

Two branches remain open, and $\mathbb{T}\,\psi \land \chi$ on line 3 remains unchecked. We apply $\land\mathbb{T}$ to it to obtain a closed tableau:

1.	$\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓				Assumption
2.	$\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓				Assumption
3.	$\mathbb{T}\,\varphi$		$\mathbb{T}\,\psi \land \chi$ ✓		$\lor\mathbb{T}\,1$
4.	$\mathbb{F}\,\varphi \lor \psi$ ✓	$\mathbb{F}\,\varphi \lor \chi$ ✓	$\mathbb{F}\,\varphi \lor \psi$ ✓	$\mathbb{F}\,\varphi \lor \chi$ ✓	$\land\mathbb{F}\,2$
5.	$\mathbb{F}\,\varphi$	$\mathbb{F}\,\varphi$	$\mathbb{F}\,\varphi$	$\mathbb{F}\,\varphi$	$\lor\mathbb{F}\,4$
6.	$\mathbb{F}\,\psi$	$\mathbb{F}\,\chi$	$\mathbb{F}\,\psi$	$\mathbb{F}\,\chi$	$\lor\mathbb{F}\,4$
7.	$\otimes$	$\otimes$	$\mathbb{T}\,\psi$	$\mathbb{T}\,\psi$	$\land\mathbb{T}\,3$
8.			$\mathbb{T}\,\chi$	$\mathbb{T}\,\chi$	$\land\mathbb{T}\,3$
			$\otimes$	$\otimes$	

For comparison, here's a closed tableau for the same set of assumptions in which the rules are applied in a different order:

$$
\begin{array}{lcccccl}
1. & \multicolumn{4}{c}{\mathbb{T}\, \varphi \lor (\psi \land \chi) \checkmark} & & \text{Assumption} \\
2. & \multicolumn{4}{c}{\mathbb{F}\, (\varphi \lor \psi) \land (\varphi \lor \chi) \checkmark} & & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{F}\, \varphi \lor \psi \checkmark} & \multicolumn{2}{c}{\mathbb{F}\, \varphi \lor \chi \checkmark} & & \land\mathbb{F}\, 2 \\
4. & \multicolumn{2}{c}{\mathbb{F}\, \varphi} & \multicolumn{2}{c}{\mathbb{F}\, \varphi} & & \lor\mathbb{F}\, 3 \\
5. & \multicolumn{2}{c}{\mathbb{F}\, \psi} & \multicolumn{2}{c}{\mathbb{F}\, \chi} & & \lor\mathbb{F}\, 3 \\
6. & \mathbb{T}\, \varphi & \mathbb{T}\, \psi \land \chi \checkmark & \mathbb{T}\, \varphi & \mathbb{T}\, \psi \land \chi \checkmark & & \lor\mathbb{T}\, 1 \\
7. & \otimes & \mathbb{T}\, \psi & \otimes & \mathbb{T}\, \psi & & \land\mathbb{T}\, 6 \\
8. & & \mathbb{T}\, \chi & & \mathbb{T}\, \chi & & \land\mathbb{T}\, 6 \\
 & & \otimes & & \otimes & &
\end{array}
$$

Problem tab.1. Give closed tableaux of the following:

1. $\mathbb{T}\, \varphi \land (\psi \land \chi), \mathbb{F}\, (\varphi \land \psi) \land \chi$.
2. $\mathbb{T}\, \varphi \lor (\psi \lor \chi), \mathbb{F}\, (\varphi \lor \psi) \lor \chi$.
3. $\mathbb{T}\, \varphi \to (\psi \to \chi), \mathbb{F}\, \psi \to (\varphi \to \chi)$.
4. $\mathbb{T}\, \varphi, \mathbb{F}\, \lnot\lnot\varphi$.

Problem tab.2. Give closed tableaux of the following:

1. $\mathbb{T}\, (\varphi \lor \psi) \to \chi, \mathbb{F}\, \varphi \to \chi$.
2. $\mathbb{T}\, (\varphi \to \chi) \land (\psi \to \chi), \mathbb{F}\, (\varphi \lor \psi) \to \chi$.
3. $\mathbb{F}\, \lnot(\varphi \land \lnot\varphi)$.
4. $\mathbb{T}\, \psi \to \varphi, \mathbb{F}\, \lnot\varphi \to \lnot\psi$.
5. $\mathbb{F}\, (\varphi \to \lnot\varphi) \to \lnot\varphi$.
6. $\mathbb{F}\, \lnot(\varphi \to \psi) \to \lnot\psi$.
7. $\mathbb{T}\, \varphi \to \chi, \mathbb{F}\, \lnot(\varphi \land \lnot\chi)$.
8. $\mathbb{T}\, \varphi \land \lnot\chi, \mathbb{F}\, \lnot(\varphi \to \chi)$.
9. $\mathbb{T}\, \varphi \lor \psi, \lnot\psi, \mathbb{F}\, \varphi$.
10. $\mathbb{T}\, \lnot\varphi \lor \lnot\psi, \mathbb{F}\, \lnot(\varphi \land \psi)$.
11. $\mathbb{F}\, (\lnot\varphi \land \lnot\psi) \to \lnot(\varphi \lor \psi)$.
12. $\mathbb{F}\, \lnot(\varphi \lor \psi) \to (\lnot\varphi \land \lnot\psi)$.

Problem tab.3. Give closed tableaux of the following:

1. $\mathbb{T}\, \lnot(\varphi \to \psi), \mathbb{F}\, \varphi$.

2. $\mathbb{T}\,\neg(\varphi \land \psi), \mathbb{F}\,\neg\varphi \lor \neg\psi$.

3. $\mathbb{T}\,\varphi \to \psi, \mathbb{F}\,\neg\varphi \lor \psi$.

4. $\mathbb{F}\,\neg\neg\varphi \to \varphi$.

5. $\mathbb{T}\,\varphi \to \psi, \mathbb{T}\,\neg\varphi \to \psi, \mathbb{F}\,\psi$.

6. $\mathbb{T}\,(\varphi \land \psi) \to \chi, \mathbb{F}\,(\varphi \to \chi) \lor (\psi \to \chi)$.

7. $\mathbb{T}\,(\varphi \to \psi) \to \varphi, \mathbb{F}\,\varphi$.

8. $\mathbb{F}\,(\varphi \to \psi) \lor (\psi \to \chi)$.

tab.6 Tableaux with Quantifiers

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Example tab.7. When dealing with quantifiers, we have to make sure not to violate the eigenvariable condition, and sometimes this requires us to play around with the order of carrying out certain inferences. In general, it helps to try and take care of rules subject to the eigenvariable condition first (they will be higher up in the finished tableau).

Let's see how we'd give a tableau for the sentence $\exists x\,\neg\varphi(x) \to \neg\forall x\,\varphi(x)$. Starting as usual, we start by recording the assumption,

1.	$\mathbb{F}\,\exists x\,\neg\varphi(x) \to \neg\forall x\,\varphi(x)$	Assumption

Since the main operator is $\to$, we apply the $\to\mathbb{F}$:

1.	$\mathbb{F}\,\exists x\,\neg\varphi(x) \to \neg\forall x\,\varphi(x)$ ✓	Assumption
2.	$\mathbb{T}\,\exists x\,\neg\varphi(x)$	$\to\mathbb{F}\,1$
3.	$\mathbb{F}\,\neg\forall x\,\varphi(x)$	$\to\mathbb{F}\,1$

The next line to deal with is 2. We use $\exists\mathbb{T}$. This requires a new constant symbol; since no constant symbols yet occur, we can pick any one, say, a.

1.	$\mathbb{F}\,\exists x\,\neg\varphi(x) \to \neg\forall x\,\varphi(x)$ ✓	Assumption
2.	$\mathbb{T}\,\exists x\,\neg\varphi(x)$ ✓	$\to\mathbb{F}\,1$
3.	$\mathbb{F}\,\neg\forall x\,\varphi(x)$	$\to\mathbb{F}\,1$
4.	$\mathbb{T}\,\neg\varphi(a)$	$\exists\mathbb{T}\,2$

Now we apply $\neg\mathbb{F}$ to line 3:

1.	$\mathbb{F}\,\exists x\,\neg\varphi(x) \to \neg\forall x\,\varphi(x)$ ✓	Assumption
2.	$\mathbb{T}\,\exists x\,\neg\varphi(x)$ ✓	$\to\mathbb{F}\,1$
3.	$\mathbb{F}\,\neg\forall x\,\varphi(x)$ ✓	$\to\mathbb{F}\,1$
4.	$\mathbb{T}\,\neg\varphi(a)$	$\exists\mathbb{T}\,2$
5.	$\mathbb{T}\,\forall x\,\varphi(x)$	$\neg\mathbb{F}\,3$

We obtain a closed tableau by applying $\neg\mathbb{T}$ to line 4, followed by $\forall\mathbb{T}$ to line 5.

1.	$\mathbb{F}\,\exists x\,\neg\varphi(x) \to \neg\forall x\,\varphi(x)$ ✓	Assumption
2.	$\mathbb{T}\,\exists x\,\neg\varphi(x)$ ✓	$\to\mathbb{F}\,1$
3.	$\mathbb{F}\,\neg\forall x\,\varphi(x)$ ✓	$\to\mathbb{F}\,1$
4.	$\mathbb{T}\,\neg\varphi(a)$	$\exists\mathbb{T}\,2$
5.	$\mathbb{T}\,\forall x\,\varphi(x)$	$\neg\mathbb{F}\,3$
6.	$\mathbb{F}\,\varphi(a)$	$\neg\mathbb{T}\,4$
7.	$\mathbb{T}\,\varphi(a)$	$\forall\mathbb{T}\,5$
	$\otimes$	

Example tab.8. Let's see how we'd give a tableau for the set

$$\mathbb{F}\,\exists x\,\chi(x,b), \mathbb{T}\,\exists x\,(\varphi(x) \land \psi(x)), \mathbb{T}\,\forall x\,(\psi(x) \to \chi(x,b)).$$

Starting as usual, we start with the assumptions:

1.	$\mathbb{F}\,\exists x\,\chi(x,b)$	Assumption
2.	$\mathbb{T}\,\exists x\,(\varphi(x) \land \psi(x))$	Assumption
3.	$\mathbb{T}\,\forall x\,(\psi(x) \to \chi(x,b))$	Assumption

We should always apply a rule with the eigenvariable condition first; in this case that would be $\exists\mathbb{T}$ to line 2. Since the assumptions contain the constant symbol b, we have to use a different one; let's pick a again.

1.	$\mathbb{F}\,\exists x\,\chi(x,b)$	Assumption
2.	$\mathbb{T}\,\exists x\,(\varphi(x) \land \psi(x))$ ✓	Assumption
3.	$\mathbb{T}\,\forall x\,(\psi(x) \to \chi(x,b))$	Assumption
4.	$\mathbb{T}\,\varphi(a) \land \psi(a)$	$\exists\mathbb{T}\,2$

If we now apply $\exists\mathbb{F}$ to line 1 or $\forall\mathbb{T}$ to line 3, we have to decide which term t to substitute for x. Since there is no eigenvariable condition for these rules, we can pick any term we like. In some cases we may even have to apply the rule several times with different ts. But as a general rule, it pays to pick one of the terms already occurring in the tableau—in this case, a and b—and in this case we can guess that a will be more likely to result in a closed branch.

1.	$\mathbb{F}\,\exists x\,\chi(x,b)$	Assumption
2.	$\mathbb{T}\,\exists x\,(\varphi(x) \land \psi(x))$ ✓	Assumption
3.	$\mathbb{T}\,\forall x\,(\psi(x) \to \chi(x,b))$	Assumption
4.	$\mathbb{T}\,\varphi(a) \land \psi(a)$	$\exists\mathbb{T}\,2$
5.	$\mathbb{F}\,\chi(a,b)$	$\exists\mathbb{F}\,1$
6.	$\mathbb{T}\,\psi(a) \to \chi(a,b)$	$\forall\mathbb{T}\,3$

We don't check the signed formulas in lines 1 and 3, since we may have to use them again. Now apply $\land\mathbb{T}$ to line 4:

1.	$\mathbb{F}\,\exists x\,\chi(x,b)$	Assumption
2.	$\mathbb{T}\,\exists x\,(\varphi(x)\land\psi(x))\ \checkmark$	Assumption
3.	$\mathbb{T}\,\forall x\,(\psi(x)\to\chi(x,b))$	Assumption
4.	$\mathbb{T}\,\varphi(a)\land\psi(a)\ \checkmark$	$\exists\mathbb{T}\,2$
5.	$\mathbb{F}\,\chi(a,b)$	$\exists\mathbb{F}\,1$
6.	$\mathbb{T}\,\psi(a)\to\chi(a,b)$	$\forall\mathbb{T}\,3$
7.	$\mathbb{T}\,\varphi(a)$	$\land\mathbb{T}\,4$
8.	$\mathbb{T}\,\psi(a)$	$\land\mathbb{T}\,4$

If we now apply $\to\mathbb{T}$ to line 6, the tableau closes:

1.	$\mathbb{F}\,\exists x\,\chi(x,b)$	Assumption
2.	$\mathbb{T}\,\exists x\,(\varphi(x)\land\psi(x))\ \checkmark$	Assumption
3.	$\mathbb{T}\,\forall x\,(\psi(x)\to\chi(x,b))$	Assumption
4.	$\mathbb{T}\,\varphi(a)\land\psi(a)\ \checkmark$	$\exists\mathbb{T}\,2$
5.	$\mathbb{F}\,\chi(a,b)$	$\exists\mathbb{F}\,1$
6.	$\mathbb{T}\,\psi(a)\to\chi(a,b)\ \checkmark$	$\forall\mathbb{T}\,3$
7.	$\mathbb{T}\,\varphi(a)$	$\land\mathbb{T}\,4$
8.	$\mathbb{T}\,\psi(a)$	$\land\mathbb{T}\,4$
9.	$\mathbb{F}\,\psi(a) \qquad \mathbb{T}\,\chi(a,b)$	$\to\mathbb{T}\,6$
	$\otimes \qquad\qquad \otimes$	

Example tab.9. We construct a tableau for the set

$$\mathbb{T}\,\forall x\,\varphi(x), \mathbb{T}\,\forall x\,\varphi(x)\to\exists y\,\psi(y), \mathbb{T}\,\neg\exists y\,\psi(y).$$

Starting as usual, we write down the assumptions:

1.	$\mathbb{T}\,\forall x\,\varphi(x)$	Assumption
2.	$\mathbb{T}\,\forall x\,\varphi(x)\to\exists y\,\psi(y)$	Assumption
3.	$\mathbb{T}\,\neg\exists y\,\psi(y)$	Assumption

We begin by applying the $\neg\mathbb{T}$ rule to line 3. A corollary to the rule "always apply rules with eigenvariable conditions first" is "defer applying quantifier rules without eigenvariable conditions until needed." Also, defer rules that result in a split.

1.	$\mathbb{T}\,\forall x\,\varphi(x)$	Assumption
2.	$\mathbb{T}\,\forall x\,\varphi(x)\to\exists y\,\psi(y)$	Assumption
3.	$\mathbb{T}\,\neg\exists y\,\psi(y)\ \checkmark$	Assumption
4.	$\mathbb{F}\,\exists y\,\psi(y)$	$\neg\mathbb{T}\,3$

The new line 4 requires $\exists\mathbb{F}$, a quantifier rule without the eigenvariable condition. So we defer this in favor of using $\to\mathbb{T}$ on line 2.

$$
\begin{array}{llcl}
1. & \multicolumn{2}{c}{\mathbb{T}\,\forall x\,\varphi(x)} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\forall x\,\varphi(x) \to \exists y\,\psi(y) \checkmark} & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{T}\,\neg\exists y\,\psi(y) \checkmark} & \text{Assumption} \\
4. & \multicolumn{2}{c}{\mathbb{F}\,\exists y\,\psi(y)} & \neg\mathbb{T}\,3 \\
5. & \mathbb{F}\,\forall x\,\varphi(x) & \mathbb{T}\,\exists y\,\psi(y) & \to\mathbb{T}\,2
\end{array}
$$

Both new signed formulas require rules with eigenvariable conditions, so these should be next:

$$
\begin{array}{llcl}
1. & \multicolumn{2}{c}{\mathbb{T}\,\forall x\,\varphi(x)} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\forall x\,\varphi(x) \to \exists y\,\psi(y) \checkmark} & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{T}\,\neg\exists y\,\psi(y) \checkmark} & \text{Assumption} \\
4. & \multicolumn{2}{c}{\mathbb{F}\,\exists y\,\psi(y)} & \neg\mathbb{T}\,3 \\
5. & \mathbb{F}\,\forall x\,\varphi(x) \checkmark & \mathbb{T}\,\exists y\,\psi(y) \checkmark & \to\mathbb{T}\,2 \\
6. & \mathbb{F}\,\varphi(b) & \mathbb{T}\,\psi(c) & \forall\mathbb{F}\,5;\ \exists\mathbb{T}\,5
\end{array}
$$

To close the branches, we have to use the signed formulas on lines 1 and 3. The corresponding rules ($\forall\mathbb{T}$ and $\exists\mathbb{F}$) don't have eigenvariable conditions, so we are free to pick whichever terms are suitable. In this case, that's b and c, respectively.

$$
\begin{array}{llcl}
1. & \multicolumn{2}{c}{\mathbb{T}\,\forall x\,\varphi(x)} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\forall x\,\varphi(x) \to \exists y\,\psi(y) \checkmark} & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{T}\,\neg\exists y\,\psi(y) \checkmark} & \text{Assumption} \\
4. & \multicolumn{2}{c}{\mathbb{F}\,\exists y\,\psi(y)} & \neg\mathbb{T}\,3 \\
5. & \mathbb{F}\,\forall x\,\varphi(x) \checkmark & \mathbb{T}\,\exists y\,\psi(y) \checkmark & \to\mathbb{T}\,2 \\
6. & \mathbb{F}\,\varphi(b) & \mathbb{T}\,\psi(c) & \forall\mathbb{F}\,5;\ \exists\mathbb{T}\,5 \\
7. & \mathbb{T}\,\varphi(b) & \mathbb{F}\,\psi(c) & \forall\mathbb{T}\,1;\ \exists\mathbb{F}\,4 \\
 & \otimes & \otimes &
\end{array}
$$

Problem tab.4. Give closed tableaux of the following:

1. $\mathbb{F}\,(\forall x\,\varphi(x) \land \forall y\,\psi(y)) \to \forall z\,(\varphi(z) \land \psi(z))$.
2. $\mathbb{F}\,(\exists x\,\varphi(x) \lor \exists y\,\psi(y)) \to \exists z\,(\varphi(z) \lor \psi(z))$.
3. $\mathbb{T}\,\forall x\,(\varphi(x) \to \psi), \mathbb{F}\,\exists y\,\varphi(y) \to \psi$.
4. $\mathbb{T}\,\forall x\,\neg\varphi(x), \mathbb{F}\,\neg\exists x\,\varphi(x)$.
5. $\mathbb{F}\,\neg\exists x\,\varphi(x) \to \forall x\,\neg\varphi(x)$.
6. $\mathbb{F}\,\neg\exists x\,\forall y\,((\varphi(x,y) \to \neg\varphi(y,y)) \land (\neg\varphi(y,y) \to \varphi(x,y)))$.

Problem tab.5. Give closed tableaux of the following:

1. $\mathbb{F}\,\neg\forall x\,\varphi(x) \to \exists x\,\neg\varphi(x)$.
2. $\mathbb{T}\,(\forall x\,\varphi(x) \to \psi), \mathbb{F}\,\exists y\,(\varphi(y) \to \psi)$.
3. $\mathbb{F}\,\exists x\,(\varphi(x) \to \forall y\,\varphi(y))$.

tab.7 Proof-Theoretic Notions

fol:tab:ptn:sec

This section collects the definitions of the provability relation and consistency for tableaux.

explanation Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions.* These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the existence of certain closed tableaux. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems.*

Definition tab.10 (Theorems). A sentence φ is a *theorem* if there is a closed tableau for $\mathbb{F}\,\varphi$. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Definition tab.11 (Derivability). A sentence φ is *derivable from* a set of sentences Γ, $\Gamma \vdash \varphi$ iff there is a finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ and a closed tableau for the set

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

If φ is not derivable from Γ we write $\Gamma \nvdash \varphi$.

Definition tab.12 (Consistency). A set of sentences Γ is *inconsistent* iff there is a finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ and a closed tableau for the set

$$\{\mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

If Γ is not inconsistent, we say it is *consistent.*

Proposition tab.13 (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.* *fol:tab:ptn:prop:reflexivity*

Proof. If $\varphi \in \Gamma$, $\{\varphi\}$ is a finite subset of Γ and the tableau

1.	$\mathbb{F}\,\varphi$	Assumption
2.	$\mathbb{T}\,\varphi$	Assumption
	$\otimes$	

is closed. □

fol:tab:ptn: prop:monotonicity **Proposition tab.14 (Monotonicity).** *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any finite subset of Γ is also a finite subset of Δ. □

fol:tab:ptn: prop:transitivity **Proposition tab.15 (Transitivity).** *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

Proof. If $\{\varphi\} \cup \Delta \vdash \psi$, then there is a finite subset $\Delta_0 = \{\chi_1, \dots, \chi_n\} \subseteq \Delta$ such that

$$\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_n\}$$

has a closed tableau. If $\Gamma \vdash \varphi$ then there are $\theta_1, \dots, \theta_m$ such that

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\theta_1, \dots, \mathbb{T}\,\theta_m\}$$

has a closed tableau.

Now consider the tableau with assumptions

$$\mathbb{F}\,\psi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_n, \mathbb{T}\,\theta_1, \dots, \mathbb{T}\,\theta_m.$$

Apply the Cut rule on φ. This generates two branches, one has $\mathbb{T}\,\varphi$ in it, the other $\mathbb{F}\,\varphi$. Thus, on the one branch, all of

$$\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_n\}$$

are available. Since there is a closed tableau for these assumptions, we can attach it to that branch; every branch through $\mathbb{T}\,\varphi$ closes. On the other branch, all of

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\theta_1, \dots, \mathbb{T}\,\theta_m\}$$

are available, so we can also complete the other side to obtain a closed tableau. This shows $\Gamma \cup \Delta \vdash \psi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i, then $\Gamma \vdash \psi$.

fol:tab:ptn: prop:incons **Proposition tab.16.** *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence φ.*

Proof. Exercise. □

Problem tab.6. Prove Proposition tab.16

fol:tab:ptn: prop:proves-compact **Proposition tab.17 (Compactness).**

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*

2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ and a closed tableau for

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

This tableau also shows $\Gamma_0 \vdash \varphi$.

2. If Γ is inconsistent, then for some finite subset $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ there is a closed tableau for

$$\{\mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

This closed tableau shows that Γ_0 is inconsistent. □

tab.8 Derivability and Consistency

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem. fol:tab:prv:sec

Proposition tab.18. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.* *fol:tab:prv:prop:provability-contr*

Proof. There are finite $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ and $\Gamma_1 = \{\chi_1, \dots, \chi_n\} \subseteq \Gamma$ such that

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$
$$\{\mathbb{T}\,\varphi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_m\}$$

have closed tableaux. Using the Cut rule on φ we can combine these into a single closed tableau that shows $\Gamma_0 \cup \Gamma_1$ is inconsistent. Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence Γ is inconsistent. □

Proposition tab.19. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.* *fol:tab:prv:prop:prov-incons*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a closed tableau for

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

Using the $\neg\mathbb{T}$ rule, this can be turned into a closed tableau for

$$\{\mathbb{T}\,\neg\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

On the other hand, if there is a closed tableau for the latter, we can turn it into a closed tableau of the former by removing every formula that results from $\neg\mathbb{T}$ applied to the first assumption $\mathbb{T}\,\neg\varphi$ as well as that assumption, and adding the assumption $\mathbb{F}\,\varphi$. For if a branch was closed before because it contained the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{T}\,\neg\varphi$, i.e., $\mathbb{F}\,\varphi$, the corresponding branch in the new tableau is also closed. If a branch in the old tableau was closed because it contained the assumption $\mathbb{T}\,\neg\varphi$ as well as $\mathbb{F}\,\neg\varphi$ we can turn it into a closed branch by applying $\neg\mathbb{F}$ to $\mathbb{F}\,\neg\varphi$ to obtain $\mathbb{T}\,\varphi$. This closes the branch since we added $\mathbb{F}\,\varphi$ as an assumption. □

Problem tab.7. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:tab:prv: prop:explicit-inc **Proposition tab.20.** *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \ldots, \mathbb{T}\,\psi_n\}$$

has a closed tableau. Replace the assumption $\mathbb{F}\,\varphi$ by $\mathbb{T}\,\neg\varphi$, and insert the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{F}\,\varphi$ after the assumptions. Any sentence in the tableau justified by appeal to line 1 in the old tableau is now justified by appeal to line $n+1$. So if the old tableau was closed, the new one is. It shows that Γ is inconsistent, since all assumptions are in Γ. □

fol:tab:prv: prop:provability-exhaustive **Proposition tab.21.** *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.*

Proof. If there are $\psi_1, \ldots, \psi_n \in \Gamma$ and $\chi_1, \ldots, \chi_m \in \Gamma$ such that

$$\{\mathbb{T}\,\varphi, \mathbb{T}\,\psi_1, \ldots, \mathbb{T}\,\psi_n\} \text{ and}$$
$$\{\mathbb{T}\,\neg\varphi, \mathbb{T}\,\chi_1, \ldots, \mathbb{T}\,\chi_m\}$$

both have closed tableaux, we can construct a single, combined tableau that shows that Γ is inconsistent by using as assumptions $\mathbb{T}\,\psi_1, \ldots, \mathbb{T}\,\psi_n$ together with $\mathbb{T}\,\chi_1, \ldots, \mathbb{T}\,\chi_m$, followed by an application of the Cut rule. This yields two branches, one starting with $\mathbb{T}\,\varphi$, the other with $\mathbb{F}\,\varphi$.

On the left left side, add the part of the first tableau below its assumptions. Here, every rule application is still correct, since each of the assumptions of the first tableau, including $\mathbb{T}\,\varphi$, is available. Thus, every branch below $\mathbb{T}\,\varphi$ closes.

On the right side, add the part of the second tableau below its assumption, with the results of any applications of $\neg\mathbb{T}$ to $\mathbb{T}\,\neg\varphi$ removed. The conclusion of $\neg\mathbb{T}$ to $\mathbb{T}\,\neg\varphi$ is $\mathbb{F}\,\varphi$, which is nevertheless available, as it is the conclusion of the Cut rule on the right side of the combined tableau.

If a branch in the second tableau was closed because it contained the assumption $\mathbb{T}\,\neg\varphi$ (which no longer appears as an assumption in the combined tableau) as well as $\mathbb{F}\,\neg\varphi$, we can applying $\neg\mathbb{F}$ to $\mathbb{F}\,\neg\varphi$ to obtain $\mathbb{T}\,\varphi$. Now the corresponding branch in the combined tableau also closes, because it contains the right-hand conclusion of the Cut rule, $\mathbb{F}\,\varphi$. If a branch in the second tableau closed for any other reason, the corresponding branch in the combined tableau also closes, since any signed formulas other than $\mathbb{T}\,\neg\varphi$ occurring on the branch in the old, second tableau also occur on the corresponding branch in the combined tableau. □

tab.9 Derivability and the Propositional Connectives

fol:tab:ppr: sec We establish that the derivability relation $\vdash$ of tableaux is strong enough to establish some basic facts involving the propositional connectives, such as that $\varphi \wedge \psi \vdash \varphi$ and $\varphi, \varphi \rightarrow \psi \vdash \psi$ (modus ponens). These facts are needed for the proof of the completeness theorem. explanation

Proposition tab.22. *fol:tab:ppr: prop:provability-land*

1. *Both* $\varphi \land \psi \vdash \varphi$ *and* $\varphi \land \psi \vdash \psi$. *fol:tab:ppr: prop:provability-land-left*
2. $\varphi, \psi \vdash \varphi \land \psi$. *fol:tab:ppr: prop:provability-land-right*

Proof. 1. Both $\{\mathbb{F}\,\varphi, \mathbb{T}\,\varphi \land \psi\}$ and $\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi \land \psi\}$ have closed tableaux

$$\begin{array}{lcl}
1. & \mathbb{F}\,\varphi & \text{Assumption} \\
2. & \mathbb{T}\,\varphi \land \psi & \text{Assumption} \\
3. & \mathbb{T}\varphi & \land\mathbb{T}\,2 \\
4. & \mathbb{T}\psi & \land\mathbb{T}\,2 \\
 & \otimes &
\end{array}$$

$$\begin{array}{lcl}
1. & \mathbb{F}\,\psi & \text{Assumption} \\
2. & \mathbb{T}\,\varphi \land \psi & \text{Assumption} \\
3. & \mathbb{T}\varphi & \land\mathbb{T}\,2 \\
4. & \mathbb{T}\psi & \land\mathbb{T}\,2 \\
 & \otimes &
\end{array}$$

2. Here is a closed tableau for $\{\mathbb{T}\,\varphi, \mathbb{T}\,\psi, \mathbb{F}\,\varphi \land \psi\}$:

$$\begin{array}{lccl}
1. & \multicolumn{2}{c}{\mathbb{F}\,\varphi \land \psi} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\varphi} & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{T}\,\psi} & \text{Assumption} \\
 & / & \backslash & \\
4. & \mathbb{F}\,\varphi & \mathbb{F}\,\psi & \land\mathbb{F}\,1 \\
 & \otimes & \otimes &
\end{array}$$

Proposition tab.23. *fol:tab:ppr: prop:provability-lor*

1. $\{\varphi \lor \psi, \lnot\varphi, \lnot\psi\}$ *is inconsistent.*
2. *Both* $\varphi \vdash \varphi \lor \psi$ *and* $\psi \vdash \varphi \lor \psi$.

Proof. 1. We give a closed tableau of $\{\mathbb{T}\,\varphi \lor \psi, \mathbb{T}\,\lnot\varphi, \mathbb{T}\,\lnot\psi\}$:

$$\begin{array}{lccl}
1. & \multicolumn{2}{c}{\mathbb{T}\,\varphi \lor \psi} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\lnot\varphi} & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{T}\,\lnot\psi} & \text{Assumption} \\
4. & \multicolumn{2}{c}{\mathbb{F}\,\varphi} & \lnot\mathbb{T}\,2 \\
5. & \multicolumn{2}{c}{\mathbb{F}\,\psi} & \lnot\mathbb{T}\,3 \\
 & / & \backslash & \\
6. & \mathbb{T}\,\varphi & \mathbb{T}\,\psi & \lor\mathbb{T}\,1 \\
 & \otimes & \otimes &
\end{array}$$

2. Both $\{\mathbb{F}\,\varphi \lor \psi, \mathbb{T}\,\varphi\}$ and $\{\mathbb{F}\,\varphi \lor \psi, \mathbb{T}\,\psi\}$ have closed tableaux:

1.	$\mathbb{F}\,\varphi \lor \psi$	Assumption
2.	$\mathbb{T}\,\varphi$	Assumption
3.	$\mathbb{F}\varphi$	$\lor\mathbb{F}\,1$
4.	$\mathbb{F}\psi$	$\lor\mathbb{F}\,1$
	$\otimes$	

1.	$\mathbb{F}\,\varphi \lor \psi$	Assumption
2.	$\mathbb{T}\,\psi$	Assumption
3.	$\mathbb{F}\varphi$	$\lor\mathbb{F}\,1$
4.	$\mathbb{F}\psi$	$\lor\mathbb{F}\,1$
	$\otimes$	

fol:tab:ppr:prop:provability-lif

Proposition tab.24.

fol:tab:ppr:prop:provability-lif-left

1. $\varphi, \varphi \to \psi \vdash \psi$.

fol:tab:ppr:prop:provability-lif-right

2. *Both* $\neg\varphi \vdash \varphi \to \psi$ *and* $\psi \vdash \varphi \to \psi$.

Proof. 1. $\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi \to \psi, \mathbb{T}\,\varphi\}$ has a closed tableau:

1.	$\mathbb{F}\,\psi$		Assumption
2.	$\mathbb{T}\,\varphi \to \psi$		Assumption
3.	$\mathbb{T}\,\varphi$		Assumption
4.	$\mathbb{F}\,\varphi$	$\mathbb{T}\,\psi$	$\to\mathbb{T}\,2$
	$\otimes$	$\otimes$	

2. Both $\{\mathbb{F}\,\varphi \to \psi, \mathbb{T}\,\neg\varphi\}$ and $\{\mathbb{F}\,\varphi \to \psi, \mathbb{T}\,\psi\}$ have closed tableaux:

1.	$\mathbb{F}\,\varphi \to \psi$	Assumption
2.	$\mathbb{T}\,\neg\varphi$	Assumption
3.	$\mathbb{T}\,\varphi$	$\to\mathbb{F}\,1$
4.	$\mathbb{F}\,\psi$	$\to\mathbb{F}\,1$
5.	$\mathbb{F}\,\varphi$	$\neg\mathbb{T}\,2$
	$\otimes$	

1.	$\mathbb{F}\,\varphi \to \psi$	Assumption
2.	$\mathbb{T}\,\psi$	Assumption
3.	$\mathbb{T}\,\varphi$	$\to\mathbb{F}\,1$
4.	$\mathbb{F}\,\psi$	$\to\mathbb{F}\,1$
	$\otimes$	

tab.10 Derivability and the Quantifiers

explanation The completeness theorem also requires that the tableaux rules yield the facts about $\vdash$ established in this section. fol:tab:qpr:sec

Theorem tab.25. *If c is a constant not occurring in Γ or $\varphi(x)$ and $\Gamma \vdash \varphi(c)$, then $\Gamma \vdash \forall x\, \varphi(x)$.* fol:tab:qpr:thm:strong-generalization

Proof. Suppose $\Gamma \vdash \varphi(c)$, i.e., there are $\psi_1, \dots, \psi_n \in \Gamma$ and a closed tableau for

$$\{\mathbb{F}\,\varphi(c), \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

We have to show that there is also a closed tableau for

$$\{\mathbb{F}\,\forall x\,\varphi(x), \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

Take the closed tableau and replace the first assumption with $\mathbb{F}\,\forall x\,\varphi(x)$, and insert $\mathbb{F}\,\varphi(c)$ after the assumptions.

$$\begin{array}{c} \mathbb{F}\,\varphi(c) \\ \mathbb{T}\,\psi_1 \\ \vdots \\ \mathbb{T}\,\psi_n \end{array} \qquad \begin{array}{c} \mathbb{F}\,\forall x\,\varphi(x) \\ \mathbb{T}\,\psi_1 \\ \vdots \\ \mathbb{T}\,\psi_n \\ \mathbb{F}\,\varphi(c) \end{array}$$

The tableau is still closed, since all sentences available as assumptions before are still available at the top of the tableau. The inserted line is the result of a correct application of $\forall\mathbb{F}$, since the constant symbol c does not occur in ψ_1, …, ψ_n or $\forall x\,\varphi(x)$, i.e., it does not occur above the inserted line in the new tableau. □

Proposition tab.26. fol:tab:qpr:prop:provability-quantifiers

1. *$\varphi(t) \vdash \exists x\,\varphi(x)$.*
2. *$\forall x\,\varphi(x) \vdash \varphi(t)$.*

Proof. 1. A closed tableau for $\mathbb{F}\,\exists x\,\varphi(x), \mathbb{T}\,\varphi(t)$ is:

1.	$\mathbb{F}\,\exists x\,\varphi(x)$	Assumption
2.	$\mathbb{T}\,\varphi(t)$	Assumption
3.	$\mathbb{F}\,\varphi(t)$	$\exists\mathbb{F}$ 1
	$\otimes$	

2. A closed tableau for $\mathbb{F}\,\varphi(t), \mathbb{T}\,\forall x\,\varphi(x)$, is:

1.	$\mathbb{F}\,\varphi(t)$	Assumption
2.	$\mathbb{T}\forall x\,\varphi(x)$	Assumption
3.	$\mathbb{T}\,\varphi(t)$	$\forall\mathbb{T}\,2$
	$\otimes$	

tab.11 Soundness

fol:tab:sou:sec A derivation system, such as tableaux, is *sound* if it cannot derive things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that explanation

1. every derivable φ is valid;
2. if a sentence is derivable from some others, it is also a consequence of them;
3. if a set of sentences is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

Because all these proof-theoretic properties are defined via closed tableaux of some kind or other, proving (1)–(3) above requires proving something about the semantic properties of closed tableaux. We will first define what it means for a signed formula to be satisfied in a structure, and then show that if a tableau is closed, no structure satisfies all its assumptions. (1)–(3) then follow as corollaries from this result.

Definition tab.27. A structure $\mathfrak{M}$ *satisfies* a signed formula $\mathbb{T}\,\varphi$ iff $\mathfrak{M} \vDash \varphi$, and it satisfies $\mathbb{F}\,\varphi$ iff $\mathfrak{M} \nvDash \varphi$. $\mathfrak{M}$ satisfies a set of signed formulas Γ iff it satisfies every $S\,\varphi \in \Gamma$. Γ is *satisfiable* if there is a structure that satisfies it, and *unsatisfiable* otherwise.

fol:tab:sou:thm:tableau-soundness **Theorem tab.28 (Soundness).** *If Γ has a closed tableau, Γ is unsatisfiable.*

Proof. Let's call a branch of a tableau satisfiable iff the set of signed formulas on it is satisfiable, and let's call a tableau satisfiable if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable tableau by one of the rules of inference always results in a satisfiable tableau. This will prove the theorem: any closed tableau results by applying rules of inference to the tableau consisting only of assumptions from Γ. So if Γ were satisfiable, any tableau for it would be satisfiable. A closed tableau, however, is clearly not satisfiable:

every branch contains both $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$, and no structure can both satisfy and not satisfy φ.

Suppose we have a satisfiable tableau, i.e., a tableau with at least one satisfiable branch. Applying a rule of inference either adds signed formulas to a branch, or splits a branch in two. If the tableau has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended tableau, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let Γ be the set of signed formulas on that branch, and let $S\,\varphi \in \Gamma$ be the signed formula to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, i.e., Γ together with the conclusions of the rule, is still satisfiable. If the rule results in a split branch, we have to show that at least one of the two resulting branches is satisfiable.

First, we consider the possible inferences that do not result in a split branch.

1. The branch is expanded by applying $\neg\mathbb{T}$ to $\mathbb{T}\,\neg\psi \in \Gamma$. Then the extended branch contains the signed formulas $\Gamma \cup \{\mathbb{F}\,\psi\}$. Suppose $\mathfrak{M} \vDash \Gamma$. In particular, $\mathfrak{M} \vDash \neg\psi$. Thus, $\mathfrak{M} \nvDash \psi$, i.e., $\mathfrak{M}$ satisfies $\mathbb{F}\,\psi$.

2. The branch is expanded by applying $\neg\mathbb{F}$ to $\mathbb{F}\,\neg\psi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\wedge\mathbb{T}$ to $\mathbb{T}\,\psi \wedge \chi \in \Gamma$, which results in two new signed formulas on the branch: $\mathbb{T}\,\psi$ and $\mathbb{T}\,\chi$. Suppose $\mathfrak{M} \vDash \Gamma$, in particular $\mathfrak{M} \vDash \psi \wedge \chi$. Then $\mathfrak{M} \vDash \psi$ and $\mathfrak{M} \vDash \chi$. This means that $\mathfrak{M}$ satisfies both $\mathbb{T}\,\psi$ and $\mathbb{T}\,\chi$.

4. The branch is expanded by applying $\vee\mathbb{F}$ to $\mathbb{F}\,\psi \vee \chi \in \Gamma$: Exercise.

5. The branch is expanded by applying $\to\mathbb{F}$ to $\mathbb{F}\,\psi \to \chi \in \Gamma$: This results in two new signed formulas on the branch: $\mathbb{T}\,\psi$ and $\mathbb{F}\,\chi$. Suppose $\mathfrak{M} \vDash \Gamma$, in particular $\mathfrak{M} \nvDash \psi \to \chi$. Then $\mathfrak{M} \vDash \psi$ and $\mathfrak{M} \nvDash \chi$. This means that $\mathfrak{M}$ satisfies both $\mathbb{T}\,\psi$ and $\mathbb{F}\,\chi$.

6. The branch is expanded by applying $\forall\mathbb{T}$ to $\mathbb{T}\,\forall x\,\psi(x) \in \Gamma$: This results in a new signed formula $\mathbb{T}\,\varphi(t)$ on the branch. Suppose $\mathfrak{M} \vDash \Gamma$, in particular, $\mathfrak{M} \vDash \forall x\,\varphi(x)$. By **??**, $\mathfrak{M} \vDash \varphi(t)$. Consequently, $\mathfrak{M}$ satisfies $\mathbb{T}\,\varphi(t)$.

7. The branch is expanded by applying $\forall\mathbb{F}$ to $\mathbb{F}\,\forall x\,\psi(x) \in \Gamma$: This results in a new signed formula $\mathbb{F}\,\varphi(a)$ where a is a constant symbol not occurring in Γ. Since Γ is satisfiable, there is a $\mathfrak{M}$ such that $\mathfrak{M} \vDash \Gamma$, in particular $\mathfrak{M} \nvDash \forall x\,\psi(x)$. We have to show that $\Gamma \cup \{\mathbb{F}\,\varphi(a)\}$ is satisfiable. To do this, we define a suitable $\mathfrak{M}'$ as follows.

   By **??**, $\mathfrak{M} \nvDash \forall x\,\psi(x)$ iff for some s, $\mathfrak{M}, s \nvDash \psi(x)$. Now let $\mathfrak{M}'$ be just like $\mathfrak{M}$, except $a^{\mathfrak{M}'} = s(x)$. By **??**, for any $\mathbb{T}\,\chi \in \Gamma$, $\mathfrak{M}' \vDash \chi$, and for any $\mathbb{F}\,\chi \in \Gamma$, $\mathfrak{M}' \nvDash \chi$, since a does not occur in Γ.

   By **??**, $\mathfrak{M}', s \nvDash \varphi(x)$. By **??**, $\mathfrak{M}', s \nvDash \varphi(a)$. Since $\varphi(a)$ is a sentence, by **??**, $\mathfrak{M}' \nvDash \varphi(a)$, i.e., $\mathfrak{M}'$ satisfies $\mathbb{F}\,\varphi(a)$.

8. The branch is expanded by applying $\exists\mathbb{T}$ to $\mathbb{T}\,\exists x\,\psi(x) \in \Gamma$: Exercise.

9. The branch is expanded by applying $\exists\mathbb{F}$ to $\mathbb{F}\,\exists x\,\psi(x) \in \Gamma$: Exercise.

Now let's consider the possible inferences that result in a split branch.

1. The branch is expanded by applying $\land\mathbb{F}$ to $\mathbb{F}\,\psi \land \chi \in \Gamma$, which results in two branches, a left one continuing through $\mathbb{F}\,\psi$ and a right one through $\mathbb{F}\,\chi$. Suppose $\mathfrak{M} \vDash \Gamma$, in particular $\mathfrak{M} \nvDash \psi \land \chi$. Then $\mathfrak{M} \nvDash \psi$ or $\mathfrak{M} \nvDash \chi$. In the former case, $\mathfrak{M}$ satisfies $\mathbb{F}\,\psi$, i.e., $\mathfrak{M}$ satisfies the formulas on the left branch. In the latter, $\mathfrak{M}$ satisfies $\mathbb{F}\,\chi$, i.e., $\mathfrak{M}$ satisfies the formulas on the right branch.

2. The branch is expanded by applying $\lor\mathbb{T}$ to $\mathbb{T}\,\psi \lor \chi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\to\mathbb{T}$ to $\mathbb{T}\,\psi \to \chi \in \Gamma$: Exercise.

4. The branch is expanded by Cut: This results in two branches, one containing $\mathbb{T}\,\psi$, the other containing $\mathbb{F}\,\psi$. Since $\mathfrak{M} \vDash \Gamma$ and either $\mathfrak{M} \vDash \psi$ or $\mathfrak{M} \nvDash \psi$, $\mathfrak{M}$ satisfies either the left or the right branch. □

Problem tab.8. Complete the proof of Theorem tab.28.

fol:tab:sou: cor:weak-soundness **Corollary tab.29.** *If $\vdash \varphi$ then φ is valid.*

fol:tab:sou: cor:entailment-soundness **Corollary tab.30.** *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.*

Proof. If $\Gamma \vdash \varphi$ then for some $\psi_1, \dots, \psi_n \in \Gamma$, $\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$ has a closed tableau. By Theorem tab.28, every structure $\mathfrak{M}$ either makes some ψ_i false or makes φ true. Hence, if $\mathfrak{M} \vDash \Gamma$ then also $\mathfrak{M} \vDash \varphi$. □

fol:tab:sou: cor:consistency-soundness **Corollary tab.31.** *If Γ is satisfiable, then it is consistent.*

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ and a closed tableau for $\{\mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$. By Theorem tab.28, there is no $\mathfrak{M}$ such that $\mathfrak{M} \vDash \psi_i$ for all $i = 1, \dots, n$. But then Γ is not satisfiable. □

tab.12 Tableaux with Identity predicate

fol:tab:ide: sec Tableaux with identity predicate require additional inference rules. The rules for $=$ are (t, t_1, and t_2 are closed terms):

$$\frac{}{\mathbb{T}\,t = t}\,= \qquad \frac{\begin{array}{c}\mathbb{T}\,t_1 = t_2\\ \mathbb{T}\,\varphi(t_1)\end{array}}{\mathbb{T}\,\varphi(t_2)}\,{=}\mathbb{T} \qquad \frac{\begin{array}{c}\mathbb{T}\,t_1 = t_2\\ \mathbb{F}\,\varphi(t_1)\end{array}}{\mathbb{F}\,\varphi(t_2)}\,{=}\mathbb{F}$$

Note that in contrast to all the other rules, $=\mathbb{T}$ and $=\mathbb{F}$ require that *two* signed formulas already appear on the branch, namely both $\mathbb{T}\, t_1 = t_2$ and $S\, \varphi(t_1)$.

Example tab.32. If s and t are closed terms, then $s = t, \varphi(s) \vdash \varphi(t)$:

$$\begin{array}{lcl} 1. & \mathbb{F}\, \varphi(t) & \text{Assumption} \\ 2. & \mathbb{T}\, s = t & \text{Assumption} \\ 3. & \mathbb{T}\, \varphi(s) & \text{Assumption} \\ 4. & \mathbb{T}\, \varphi(t) & =\mathbb{T}\, 2, 3 \\ & \otimes & \end{array}$$

This may be familiar as the principle of substitutability of identicals, or Leibniz' Law.

Tableaux prove that $=$ is symmetric, i.e., that $s_1 = s_2 \vdash s_2 = s_1$:

$$\begin{array}{lcl} 1. & \mathbb{F}\, s_2 = s_1 & \text{Assumption} \\ 2. & \mathbb{T}\, s_1 = s_2 & \text{Assumption} \\ 3. & \mathbb{T}\, s_1 = s_1 & = \\ 4. & \mathbb{T}\, s_2 = s_1 & =\mathbb{T}\, 2, 3 \\ & \otimes & \end{array}$$

Here, line 2 is the first prerequisite formula $\mathbb{T}\, s_1 = s_2$ of $=\mathbb{T}$. Line 3 is the second one, of the form $\mathbb{T}\, \varphi(s_2)$—think of $\varphi(x)$ as $x = s_1$, then $\varphi(s_1)$ is $s_1 = s_1$ and $\varphi(s_2)$ is $s_2 = s_1$.

They also prove that $=$ is transitive, i.e., that $s_1 = s_2, s_2 = s_3 \vdash s_1 = s_3$:

$$\begin{array}{lcl} 1. & \mathbb{F}\, s_1 = s_3 & \text{Assumption} \\ 2. & \mathbb{T}\, s_1 = s_2 & \text{Assumption} \\ 3. & \mathbb{T}\, s_2 = s_3 & \text{Assumption} \\ 4. & \mathbb{T}\, s_1 = s_3 & =\mathbb{T}\, 3, 2 \\ & \otimes & \end{array}$$

In this tableau, the first prerequisite formula of $=\mathbb{T}$ is line 3, $\mathbb{T}\, s_2 = s_3$ (s_2 plays the role of t_1, and s_3 the role of t_2). The second prerequisite, of the form $\mathbb{T}\, \varphi(s_2)$ is line 2. Here, think of $\varphi(x)$ as $s_1 = x$; that makes $\varphi(s_2)$ into $t_1 = t_2$ (i.e., line 2) and $\varphi(s_3)$ into the formula $s_1 = s_3$ in the conclusion.

Problem tab.9. Give closed tableaux for the following:

1. $\mathbb{F}\, \forall x\, \forall y\, ((x = y \land \varphi(x)) \to \varphi(y))$

2. $\mathbb{F}\, \exists x\, (\varphi(x) \land \forall y\, (\varphi(y) \to y = x))$,
   $\mathbb{T}\, \exists x\, \varphi(x) \land \forall y\, \forall z\, ((\varphi(y) \land \varphi(z)) \to y = z)$

tab.13 Soundness with Identity predicate

fol:tab:sid:sec

Proposition tab.33. *Tableaux with rules for identity are sound: no closed tableau is satisfiable.*

Proof. We just have to show as before that if a tableau has a satisfiable branch, the branch resulting from applying one of the rules for $=$ to it is also satisfiable. Let Γ be the set of signed formulas on the branch, and let $\mathfrak{M}$ be a structure satisfying Γ.

Suppose the branch is expanded using $=$, i.e., by adding the signed formula $\mathbb{T}\, t = t$. Trivially, $\mathfrak{M} \vDash t = t$, so $\mathfrak{M}$ also satisfies $\Gamma \cup \{\mathbb{T}\, t = t\}$.

If the branch is expanded using $=\mathbb{T}$, we add a signed formula $S\, \varphi(t_2)$, but Γ contains both $\mathbb{T}\, t_1 = t_2$ and $\mathbb{T}\, \varphi(t_1)$. Thus we have $\mathfrak{M} \vDash t_1 = t_2$ and $\mathfrak{M} \vDash \varphi(t_1)$. Let s be a variable assignment with $s(x) = \mathrm{Val}^{\mathfrak{M}}(t_1)$. By **??**, $\mathfrak{M}, s \vDash \varphi(t_1)$. Since $s \sim_x s$, by **??**, $\mathfrak{M}, s \vDash \varphi(x)$. since $\mathfrak{M} \vDash t_1 = t_2$, we have $\mathrm{Val}^{\mathfrak{M}}(t_1) = \mathrm{Val}^{\mathfrak{M}}(t_2)$, and hence $s(x) = \mathrm{Val}^{\mathfrak{M}}(t_2)$. By applying **??** again, we also have $\mathfrak{M}, s \vDash \varphi(t_2)$. By **??**, $\mathfrak{M} \vDash \varphi(t_2)$. The case of $=\mathbb{F}$ is treated similarly. □

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Bibliography