tab.1 Soundness

fol:tab:sou:sec

A derivation system, such as tableaux, is *sound* if it cannot derive things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that explanation

1. every derivable φ is valid;
2. if a sentence is derivable from some others, it is also a consequence of them;
3. if a set of sentences is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

Because all these proof-theoretic properties are defined via closed tableaux of some kind or other, proving (1)–(3) above requires proving something about the semantic properties of closed tableaux. We will first define what it means for a signed formula to be satisfied in a structure, and then show that if a tableau is closed, no structure satisfies all its assumptions. (1)–(3) then follow as corollaries from this result.

Definition tab.1. A structure $\mathfrak{M}$ *satisfies* a signed formula $\mathbb{T}\,\varphi$ iff $\mathfrak{M} \vDash \varphi$, and it satisfies $\mathbb{F}\,\varphi$ iff $\mathfrak{M} \nvDash \varphi$. $\mathfrak{M}$ satisfies a set of signed formulas Γ iff it satisfies every $S\,\varphi \in \Gamma$. Γ is *satisfiable* if there is a structure that satisfies it, and *unsatisfiable* otherwise.

fol:tab:sou:thm:tableau-soundness

Theorem tab.2 (Soundness)**.** *If Γ has a closed tableau, Γ is unsatisfiable.*

Proof. Let's call a branch of a tableau satisfiable iff the set of signed formulas on it is satisfiable, and let's call a tableau satisfiable if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable tableau by one of the rules of inference always results in a satisfiable tableau. This will prove the theorem: any closed tableau results by applying rules of inference to the tableau consisting only of assumptions from Γ. So if Γ were satisfiable, any tableau for it would be satisfiable. A closed tableau, however, is clearly not satisfiable: every branch contains both $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$, and no structure can both satisfy and not satisfy φ.

Suppose we have a satisfiable tableau, i.e., a tableau with at least one satisfiable branch. Applying a rule of inference either adds signed formulas to a branch, or splits a branch in two. If the tableau has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended tableau, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let Γ be the set of signed formulas on that branch, and let $S\,\varphi \in \Gamma$ be the signed formula to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, i.e., Γ together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable.

First, we consider the possible inferences with only one premise.

1. The branch is expanded by applying $\neg\mathbb{T}$ to $\mathbb{T}\,\neg\psi \in \Gamma$. Then the extended branch contains the signed formulas $\Gamma \cup \{\mathbb{F}\,\psi\}$. Suppose $\mathfrak{M} \vDash \Gamma$. In particular, $\mathfrak{M} \vDash \neg\psi$. Thus, $\mathfrak{M} \nvDash \psi$, i.e., $\mathfrak{M}$ satisfies $\mathbb{F}\,\psi$.

2. The branch is expanded by applying $\neg\mathbb{F}$ to $\mathbb{F}\,\neg\psi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\wedge\mathbb{T}$ to $\mathbb{T}\,\psi \wedge \chi \in \Gamma$, which results in two new signed formulas on the branch: $\mathbb{T}\,\psi$ and $\mathbb{T}\,\chi$. Suppose $\mathfrak{M} \vDash \Gamma$, in particular $\mathfrak{M} \vDash \psi \wedge \chi$. Then $\mathfrak{M} \vDash \psi$ and $\mathfrak{M} \vDash \chi$. This means that $\mathfrak{M}$ satisfies both $\mathbb{T}\,\psi$ and $\mathbb{T}\,\chi$.

4. The branch is expanded by applying $\vee\mathbb{F}$ to $\mathbb{T}\,\psi \vee \chi \in \Gamma$: Exercise.

5. The branch is expanded by applying $\rightarrow\mathbb{F}$ to $\mathbb{T}\,\psi \rightarrow \chi \in \Gamma$: This results in two new signed formulas on the branch: $\mathbb{T}\,\psi$ and $\mathbb{F}\,\chi$. Suppose $\mathfrak{M} \vDash \Gamma$, in particular $\mathfrak{M} \nvDash \psi \rightarrow \chi$. Then $\mathfrak{M} \vDash \psi$ and $\mathfrak{M} \nvDash \chi$. This means that $\mathfrak{M}$ satisfies both $\mathbb{T}\,\psi$ and $\mathbb{F}\,\chi$.

6. The branch is expanded by applying $\forall\mathbb{T}$ to $\mathbb{T}\,\forall x\,\psi(x) \in \Gamma$: This results in a new signed formula $\mathbb{T}\,\varphi(t)$ on the branch. Suppose $\mathfrak{M} \vDash \Gamma$, in particular, $\mathfrak{M} \vDash \forall x\,\varphi(x)$. By **??**, $\mathfrak{M} \vDash \varphi(t)$. Consequently, $\mathfrak{M}$ satisfies $\mathbb{T}\,\varphi(t)$.

7. The branch is expanded by applying $\forall\mathbb{F}$ to $\mathbb{F}\,\forall x\,\psi(x) \in \Gamma$: This results in a new signed formula $\mathbb{F}\,\varphi(a)$ where a is a constant symbol not occurring in Γ. Since Γ is satisfiable, there is a $\mathfrak{M}$ such that $\mathfrak{M} \vDash \Gamma$, in particular $\mathfrak{M} \nvDash \forall x\,\psi(x)$. We have to show that $\Gamma \cup \{\mathbb{F}\,\varphi(a)\}$ is satisfiable. To do this, we define a suitable $\mathfrak{M}'$ as follows.

   By **??**, $\mathfrak{M} \nvDash \forall x\,\psi(x)$ iff for some s, $\mathfrak{M}, s \nvDash \psi(x)$. Now let $\mathfrak{M}'$ be just like $\mathfrak{M}$, except $a^{\mathfrak{M}'} = s(x)$. By **??**, for any $\mathbb{T}\,\chi \in \Gamma$, $\mathfrak{M}' \vDash \chi$, and for any $\mathbb{F}\,\chi \in \Gamma$, $\mathfrak{M}' \nvDash \chi$, since a does not occur in Γ.

   By **??**, $\mathfrak{M}', s \nvDash \varphi(x)$. By **??**, $\mathfrak{M}', s \nvDash \varphi(a)$. Since $\varphi(a)$ is a sentence, by **??**, $\mathfrak{M}' \nvDash \varphi(a)$, i.e., $\mathfrak{M}'$ satisfies $\mathbb{F}\,\varphi(a)$.

8. The branch is expanded by applying $\exists\mathbb{T}$ to $\mathbb{T}\,\exists x\,\psi(x) \in \Gamma$: Exercise.

9. The branch is expanded by applying $\exists\mathbb{F}$ to $\mathbb{F}\,\exists x\,\psi(x) \in \Gamma$: Exercise.

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying $\wedge\mathbb{F}$ to $\mathbb{F}\,\psi \wedge \chi \in \Gamma$, which results in two branches, a left one continuing through $\mathbb{F}\,\psi$ and a right one through $\mathbb{F}\,\chi$. Suppose $\mathfrak{M} \vDash \Gamma$, in particular $\mathfrak{M} \nvDash \psi \wedge \chi$. Then $\mathfrak{M} \nvDash \psi$ or $\mathfrak{M} \nvDash \chi$.

In the former case, $\mathfrak{M}$ satisfies $\mathbb{F}\,\psi$, i.e., $\mathfrak{M}$ satisfies the formulas on the left branch. In the latter, $\mathfrak{M}$ satisfies $\mathbb{F}\,\chi$, i.e., $\mathfrak{M}$ satisfies the formulas on the right branch.

2. The branch is expanded by applying $\vee\mathbb{T}$ to $\mathbb{T}\,\psi \vee \chi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\to\mathbb{T}$ to $\mathbb{T}\,\psi \to \chi \in \Gamma$: Exercise.

4. The branch is expanded by Cut: This results in two branches, one containing $\mathbb{T}\,\psi$, the other containing $\mathbb{F}\,\psi$. Since $\mathfrak{M} \vDash \Gamma$ and either $\mathfrak{M} \vDash \psi$ or $\mathfrak{M} \nvDash \psi$, $\mathfrak{M}$ satisfies either the left or the right branch.

$\square$

Problem tab.1. Complete the proof of Theorem tab.2.

fol:tab:sou: cor:weak-soundness

Corollary tab.3. *If* $\vdash \varphi$ *then* φ *is valid.*

fol:tab:sou: cor:entailment-soundness

Corollary tab.4. *If* $\Gamma \vdash \varphi$ *then* $\Gamma \vDash \varphi$*.*

Proof. If $\Gamma \vdash \varphi$ then for some $\psi_1, \dots, \psi_n \in \Gamma$, $\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$ has a closed tableau. By Theorem tab.2, every structure $\mathfrak{M}$ either makes some ψ_i false or makes φ true. Hence, if $\mathfrak{M} \vDash \Gamma$ then also $\mathfrak{M} \vDash \varphi$. $\square$

fol:tab:sou: cor:consistency-soundness

Corollary tab.5. *If* Γ *is satisfiable, then it is consistent.*

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ and a closed tableau for $\{\mathbb{T}\,\psi, \dots, \mathbb{T}\,\psi\}$. By Theorem tab.2, there is no $\mathfrak{M}$ such that $\mathfrak{M} \vDash \psi_i$ for all $i = 1, \dots, n$. But then Γ is not satisfiable. $\square$

Photo Credits

Bibliography