Proposition tab.1. Tableaux with rules for identity are sound: no closed tableau is satisfiable.

Proof. We just have to show as before that if a tableau has a satisfiable branch, the branch resulting from applying one of the rules for = to it is also satisfiable. Let $\Gamma$ be the set of signed formulas on the branch, and let $\mathfrak{M}$ be a structure satisfying $\Gamma$.

Suppose the branch is expanded using $=\top$, i.e., by adding the signed formula $\top t = t$. Trivially, $\mathfrak{M} \models t = t$, so $\mathfrak{M}$ also satisfies $\Gamma \cup \{\top t = t\}$.

If the branch is expanded using $=\top$, we add a signed formula $\top \varphi(t_2)$, but $\Gamma$ contains both $\top t_1 = t_2$ and $\top \varphi(t_1)$. Thus we have $\mathfrak{M} \models t_1 = t_2$ and $\mathfrak{M} \models \varphi(t_1)$. Let $s$ be a variable assignment with $s(x) = \text{Val}_{\mathfrak{M}}(t_1)$. By $\top$, $\mathfrak{M}, s \models \varphi(t_1)$. Since $s \sim x s$, by $\top$, $\mathfrak{M}, s \models \varphi(x)$. Since $\mathfrak{M} \models t_1 = t_2$, we have $\text{Val}_{\mathfrak{M}}(t_1) = \text{Val}_{\mathfrak{M}}(t_2)$, and hence $s(x) = \text{Val}_{\mathfrak{M}}(t_2)$. By applying $\top$ again, we also have $\mathfrak{M}, s \models \varphi(t_2)$. By $\top$, $\mathfrak{M} \models \varphi(t_2)$. The case of $=\bot$ is treated similarly. □

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Bibliography