

## tab.1 Examples of Tableaux

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**Example tab.1.** Let's find a closed **tableau** for the **sentence**  $(\varphi \wedge \psi) \rightarrow \varphi$ .

We begin by writing the corresponding assumption at the top of the **tableau**.

1.  $\mathbb{F}(\varphi \wedge \psi) \rightarrow \varphi$  Assumption

There is only one assumption, so only one **signed formula** to which we can apply a rule. (For every **signed formula**, there is always at most one rule that can be applied: it's the rule for the corresponding sign and **main operator** of the **sentence**.) In this case, this means, we must apply  $\rightarrow\mathbb{F}$ .

1.  $\mathbb{F}(\varphi \wedge \psi) \rightarrow \varphi \checkmark$  Assumption
2.  $\mathbb{T}\varphi \wedge \psi$   $\rightarrow\mathbb{F}$  1
3.  $\mathbb{F}\varphi$   $\rightarrow\mathbb{F}$  1

To keep track of which **signed formulas** we have applied their corresponding rules to, we write a checkmark next to the sentence. However, *only* write a checkmark if the rule has been applied to all open branches. Once a **signed formula** has had the corresponding rule applied in every open branch, we will not have to return to it and apply the rule again. (In this case, there is only one branch, so the rule only has to be applied once.)

There is one new **signed formula** to which we can apply a rule: the  $\mathbb{T}\varphi \wedge \psi$  on line 3. Applying the  $\wedge\mathbb{T}$  rule results in:

1.  $\mathbb{F}(\varphi \wedge \psi) \rightarrow \varphi \checkmark$  Assumption
  2.  $\mathbb{T}\varphi \wedge \psi \checkmark$   $\rightarrow\mathbb{F}$  1
  3.  $\mathbb{F}\varphi$   $\rightarrow\mathbb{F}$  1
  4.  $\mathbb{T}\varphi$   $\wedge\mathbb{T}$  2
  5.  $\mathbb{T}\psi$   $\wedge\mathbb{T}$  2
- $\otimes$

Since the branch now contains both  $\mathbb{T}\varphi$  (on line 4) and  $\mathbb{F}\varphi$  (on line 3), the branch is closed. Since it is the only branch, the **tableau** is closed. We have found a closed **tableau** for  $(\varphi \wedge \psi) \rightarrow \varphi$ .

**Example tab.2.** Now let's find a closed **tableau** for  $(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$ .

We begin with the corresponding assumption:

1.  $\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$  Assumption

The one **signed formula** in this **tableau** has **main operator**  $\rightarrow$  and sign  $\mathbb{F}$ , so we apply the  $\rightarrow\mathbb{F}$  rule to it to obtain:

1.  $\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$  Assumption
2.  $\mathbb{T}\neg\varphi \vee \psi$   $\rightarrow\mathbb{F}$  1
3.  $\mathbb{F}(\varphi \rightarrow \psi)$   $\rightarrow\mathbb{F}$  1

We now have a choice as to whether to apply  $\vee\text{T}$  to line 2 or  $\rightarrow\text{F}$  to line 3. It actually doesn't matter which order we pick, as long as each **signed formula** has its corresponding rule applied in every branch. So let's pick the first one. The  $\vee\text{T}$  rule allows the **tableau** to branch, and the two conclusions of the rule will be the new **signed formulas** added to the two new branches. This results in:

1.	$\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$	Assumption
2.	$\mathbb{T}\neg\varphi \vee \psi \checkmark$	$\rightarrow\text{F } 1$
3.	$\mathbb{F}(\varphi \rightarrow \psi)$	$\rightarrow\text{F } 1$
$\swarrow \quad \searrow$		
4.	$\mathbb{T}\neg\varphi \quad \mathbb{T}\psi$	$\vee\text{T } 2$

We have not applied the  $\rightarrow\text{F}$  rule to line 3 yet: let's do that now. To save time, we apply it to both branches. Recall that we write a checkmark next to a **signed formula** only if we have applied the corresponding rule in every open branch. So it's a good idea to apply a rule at the end of every branch that contains the **signed formula** the rule applies to. That way we won't have to return to that **signed formula** lower down in the various branches.

1.	$\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$	Assumption
2.	$\mathbb{T}\neg\varphi \vee \psi \checkmark$	$\rightarrow\text{F } 1$
3.	$\mathbb{F}(\varphi \rightarrow \psi) \checkmark$	$\rightarrow\text{F } 1$
$\swarrow \quad \searrow$		
4.	$\mathbb{T}\neg\varphi \quad \mathbb{T}\psi$	$\vee\text{T } 2$
5.	$\mathbb{T}\varphi \quad \mathbb{T}\varphi$	$\rightarrow\text{F } 3$
6.	$\mathbb{F}\psi \quad \mathbb{F}\psi$	$\rightarrow\text{F } 3$
$\otimes$		

The right branch is now closed. On the left branch, we can still apply the  $\neg\text{T}$  rule to line 4. This results in  $\mathbb{F}\varphi$  and closes the left branch:

1.	$\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$	Assumption
2.	$\mathbb{T}\neg\varphi \vee \psi \checkmark$	$\rightarrow\text{F } 1$
3.	$\mathbb{F}(\varphi \rightarrow \psi) \checkmark$	$\rightarrow\text{F } 1$
$\swarrow \quad \searrow$		
4.	$\mathbb{T}\neg\varphi \quad \mathbb{T}\psi$	$\vee\text{T } 2$
5.	$\mathbb{T}\varphi \quad \mathbb{T}\varphi$	$\rightarrow\text{F } 3$
6.	$\mathbb{F}\psi \quad \mathbb{F}\psi$	$\rightarrow\text{F } 3$
7.	$\mathbb{F}\varphi \quad \otimes$	$\neg\text{T } 4$
$\otimes$		

**Example tab.3.** We can give **tableaux** for any number of **signed formulas** as assumptions. Often it is also necessary to apply more than one rule that allows branching; and in general a **tableau** can have any number of branches. For

instance, consider a **tableau** for  $\{\mathbb{T}\varphi \vee (\psi \wedge \chi), \mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi)\}$ . We start by applying the  $\vee\mathbb{T}$  to the first assumption:

1.	$\mathbb{T}\varphi \vee (\psi \wedge \chi) \checkmark$	Assumption
2.	$\mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi)$	Assumption
$\swarrow \quad \searrow$		
3.	$\mathbb{T}\varphi \quad \mathbb{T}\psi \wedge \chi$	$\vee\mathbb{T} 1$

Now we can apply the  $\wedge\mathbb{F}$  rule to line 2. We do this on both branches simultaneously, and can therefore check off line 2:

1.	$\mathbb{T}\varphi \vee (\psi \wedge \chi) \checkmark$	Assumption
2.	$\mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$	Assumption
$\swarrow \quad \searrow$		
3.	$\mathbb{T}\varphi \quad \mathbb{T}\psi \wedge \chi$	$\vee\mathbb{T} 1$
$\swarrow \quad \searrow \quad \swarrow \quad \searrow$		
4.	$\mathbb{F}\varphi \vee \psi \quad \mathbb{F}\varphi \vee \chi \quad \mathbb{F}\varphi \vee \psi \quad \mathbb{F}\varphi \vee \chi$	$\wedge\mathbb{F} 2$

Now we can apply  $\vee\mathbb{F}$  to all the branches containing  $\varphi \vee \psi$ :

1.	$\mathbb{T}\varphi \vee (\psi \wedge \chi) \checkmark$	Assumption
2.	$\mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$	Assumption
$\swarrow \quad \searrow$		
3.	$\mathbb{T}\varphi \quad \mathbb{T}\psi \wedge \chi$	$\vee\mathbb{T} 1$
$\swarrow \quad \searrow \quad \swarrow \quad \searrow$		
4.	$\mathbb{F}\varphi \vee \psi \checkmark \quad \mathbb{F}\varphi \vee \chi \quad \mathbb{F}\varphi \vee \psi \checkmark \quad \mathbb{F}\varphi \vee \chi$	$\wedge\mathbb{F} 2$
5.	$\mathbb{F}\varphi \quad \mathbb{F}\varphi$	$\vee\mathbb{F} 4$
6.	$\mathbb{F}\psi \quad \mathbb{F}\psi$	$\vee\mathbb{F} 4$
$\otimes$		

The leftmost branch is now closed. Let's now apply  $\vee\mathbb{F}$  to  $\varphi \vee \chi$ :

1.	$\mathbb{T}\varphi \vee (\psi \wedge \chi) \checkmark$	Assumption
2.	$\mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$	Assumption
$\swarrow \quad \searrow$		
3.	$\mathbb{T}\varphi \quad \mathbb{T}\psi \wedge \chi$	$\vee\mathbb{T} 1$
$\swarrow \quad \searrow \quad \swarrow \quad \searrow$		
4.	$\mathbb{F}\varphi \vee \psi \checkmark \quad \mathbb{F}\varphi \vee \chi \checkmark \quad \mathbb{F}\varphi \vee \psi \checkmark \quad \mathbb{F}\varphi \vee \chi \checkmark$	$\wedge\mathbb{F} 2$
5.	$\mathbb{F}\varphi \quad \mathbb{F}\varphi$	$\vee\mathbb{F} 4$
6.	$\mathbb{F}\psi \quad \mathbb{F}\psi$	$\vee\mathbb{F} 4$
7.	$\otimes \quad \mathbb{F}\varphi \quad \mathbb{F}\varphi$	$\vee\mathbb{F} 4$
8.	$\quad \mathbb{F}\chi \quad \mathbb{F}\chi$	$\vee\mathbb{F} 4$
$\otimes$		

Note that we moved the result of applying  $\vee\mathbb{F}$  a second time below for clarity. In this instance it would not have been needed, since the justifications would have been the same.

Two branches remain open, and  $\mathbb{T}\psi \wedge \chi$  on line 3 remains unchecked. We apply  $\wedge\mathbb{T}$  to it to obtain a closed **tableau**:

1.	$\mathbb{T}\varphi \vee (\psi \wedge \chi) \checkmark$				Assumption
2.	$\mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$				Assumption
3.	$\mathbb{T}\varphi$		$\mathbb{T}\psi \wedge \chi \checkmark$		$\vee\mathbb{T} 1$
4.	$\mathbb{F}\varphi \vee \psi \checkmark$	$\mathbb{F}\varphi \vee \chi \checkmark$	$\mathbb{F}\varphi \vee \psi \checkmark$	$\mathbb{F}\varphi \vee \chi \checkmark$	$\wedge\mathbb{F} 2$
5.	$\mathbb{F}\varphi$	$\mathbb{F}\varphi$	$\mathbb{F}\varphi$	$\mathbb{F}\varphi$	$\vee\mathbb{F} 4$
6.	$\mathbb{F}\psi$	$\mathbb{F}\chi$	$\mathbb{F}\psi$	$\mathbb{F}\chi$	$\vee\mathbb{F} 4$
7.	$\otimes$	$\otimes$	$\mathbb{T}\psi$	$\mathbb{T}\psi$	$\wedge\mathbb{T} 3$
8.			$\mathbb{T}\chi$	$\mathbb{T}\chi$	$\wedge\mathbb{T} 3$
			$\otimes$	$\otimes$	

For comparison, here's a closed **tableau** for the same set of assumptions in which the rules are applied in a different order:

1.	$\mathbb{T}\varphi \vee (\psi \wedge \chi) \checkmark$				Assumption
2.	$\mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$				Assumption
3.	$\mathbb{F}\varphi \vee \psi \checkmark$		$\mathbb{F}\varphi \vee \chi \checkmark$		$\wedge\mathbb{F} 2$
4.	$\mathbb{F}\varphi$		$\mathbb{F}\varphi$		$\vee\mathbb{F} 3$
5.	$\mathbb{F}\psi$		$\mathbb{F}\chi$		$\vee\mathbb{F} 3$
6.	$\mathbb{T}\varphi$	$\mathbb{T}\psi \wedge \chi \checkmark$	$\mathbb{T}\varphi$	$\mathbb{T}\psi \wedge \chi \checkmark$	$\vee\mathbb{T} 1$
7.	$\otimes$	$\mathbb{T}\psi$	$\otimes$	$\mathbb{T}\psi$	$\wedge\mathbb{T} 3$
8.		$\mathbb{T}\chi$		$\mathbb{T}\chi$	$\wedge\mathbb{T} 3$
		$\otimes$		$\otimes$	

**Problem tab.1.** Give closed **tableaux** of the following:

1.  $\mathbb{F}\neg(\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \neg\psi)$
2.  $\mathbb{F}(\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi), \mathbb{T}(\varphi \wedge \psi) \rightarrow \chi$

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## Bibliography