

## tab.1 Derivability and Consistency

fol:tab:prv:  
sec We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:tab:prv:  
prop:provability-contr **Proposition tab.1.** *If  $\Gamma \vdash \varphi$  and  $\Gamma \cup \{\varphi\}$  is inconsistent, then  $\Gamma$  is inconsistent.*

*Proof.* There are finite  $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$  and  $\Gamma_1 = \{\chi_1, \dots, \chi_m\} \subseteq \Gamma$  such that

$$\begin{aligned} & \{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\} \\ & \{\mathbb{T} \neg \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_m\} \end{aligned}$$

have closed **tableaux**. Using the Cut rule on  $\varphi$  we can combine these into a single closed **tableau** that shows  $\Gamma_0 \cup \Gamma_1$  is inconsistent. Since  $\Gamma_0 \subseteq \Gamma$  and  $\Gamma_1 \subseteq \Gamma$ ,  $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$ , hence  $\Gamma$  is inconsistent.  $\square$

fol:tab:prv:  
prop:prov-incons **Proposition tab.2.**  *$\Gamma \vdash \varphi$  iff  $\Gamma \cup \{\neg \varphi\}$  is inconsistent.*

*Proof.* First suppose  $\Gamma \vdash \varphi$ , i.e., there is a closed **tableau** for

$$\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}$$

Using the  $\neg\mathbb{T}$  rule, this can be turned into a closed **tableau** for

$$\{\mathbb{T} \neg \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}.$$

On the other hand, if there is a closed **tableau** for the latter, we can turn it into a closed **tableau** of the former by removing every formula that results from  $\neg\mathbb{T}$  applied to the first assumption  $\mathbb{T} \neg \varphi$  as well as that assumption, and adding the assumption  $\mathbb{F} \varphi$ . For if a branch was closed before because it contained the conclusion of  $\neg\mathbb{T}$  applied to  $\mathbb{T} \neg \varphi$ , i.e.,  $\mathbb{F} \varphi$ , the corresponding branch in the new **tableau** is also closed. If a branch in the old tableau was closed because it contained the assumption  $\mathbb{T} \neg \varphi$  as well as  $\mathbb{F} \neg \varphi$  we can turn it into a closed branch by applying  $\neg\mathbb{F}$  to  $\mathbb{F} \neg \varphi$  to obtain  $\mathbb{T} \varphi$ . This closes the branch since we added  $\mathbb{F} \varphi$  as an assumption.  $\square$

**Problem tab.1.** Prove that  $\Gamma \vdash \neg \varphi$  iff  $\Gamma \cup \{\varphi\}$  is inconsistent.

fol:tab:prv:  
prop:explicit-inc **Proposition tab.3.** *If  $\Gamma \vdash \varphi$  and  $\neg \varphi \in \Gamma$ , then  $\Gamma$  is inconsistent.*

*Proof.* Suppose  $\Gamma \vdash \varphi$  and  $\neg \varphi \in \Gamma$ . Then there are  $\psi_1, \dots, \psi_n \in \Gamma$  such that

$$\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}$$

has a closed tableau. Replace the assumption  $\mathbb{F} \varphi$  by  $\mathbb{T} \neg \varphi$ , and insert the conclusion of  $\neg\mathbb{T}$  applied to  $\mathbb{F} \varphi$  after the assumptions. Any **sentence** in the **tableau** justified by appeal to line 1 in the old **tableau** is now justified by appeal to line  $n + 1$ . So if the old **tableau** was closed, the new one is. It shows that  $\Gamma$  is inconsistent, since all assumptions are in  $\Gamma$ .  $\square$

**Proposition tab.4.** *If  $\Gamma \cup \{\varphi\}$  and  $\Gamma \cup \{\neg\varphi\}$  are both inconsistent, then  $\Gamma$  is inconsistent.* fol.tab:prv:  
prop:provability-exhaustive

*Proof.* If there are  $\psi_1, \dots, \psi_n \in \Gamma$  and  $\chi_1, \dots, \chi_m \in \Gamma$  such that

$$\begin{aligned} & \{\mathbb{T}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\} \\ & \{\mathbb{T}\neg\varphi, \mathbb{T}\chi_1, \dots, \mathbb{T}\chi_m\} \end{aligned}$$

both have closed **tableaux**, we can construct a **tableau** that shows that  $\Gamma$  is inconsistent by using as assumptions  $\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n$  together with  $\mathbb{T}\chi_1, \dots, \mathbb{T}\chi_m$ , followed by an application of the Cut rule, yielding two branches, one starting with  $\mathbb{T}\varphi$ , the other with  $\mathbb{F}\varphi$ . Add on the part below the assumptions of the first **tableau** on the left side. Here, every rule application is still correct, and every branch closes. On the right side, add the part below the assumptions of the second **tableau**, with the results of any applications of  $\neg\mathbb{T}$  to  $\mathbb{T}\neg\varphi$  removed.

For if a branch was closed before because it contained the conclusion of  $\neg\mathbb{T}$  applied to  $\mathbb{T}\neg\varphi$ , i.e.,  $\mathbb{F}\varphi$ , as well as  $\mathbb{F}\varphi$ , the corresponding branch in the new **tableau** is also closed. If a branch in the old tableau was closed because it contained the assumption  $\mathbb{T}\neg\varphi$  as well as  $\mathbb{F}\neg\varphi$  we can turn it into a closed branch by applying  $\neg\mathbb{F}$  to  $\mathbb{F}\neg\varphi$  to obtain  $\mathbb{T}\varphi$ . □

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## Bibliography