tab.1 Derivability and Consistency

fol:tab:prv: sec

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:tab:prv: prop:provability-contr

Proposition tab.1. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Proof. There are finite $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ and $\Gamma_1 = \{\chi_1, \dots, \chi_n\} \subseteq \Gamma$ such that

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$
$$\{\mathbb{T}\,\neg\varphi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_m\}$$

have closed tableaux. Using the Cut rule on φ we can combine these into a single closed tableau that shows $\Gamma_0 \cup \Gamma_1$ is inconsistent. Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence Γ is inconsistent. □

fol:tab:prv: prop:prov-incons

Proposition tab.2. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a closed tableau for

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

Using the $\neg\mathbb{T}$ rule, this can be turned into a closed tableau for

$$\{\mathbb{T}\,\neg\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

On the other hand, if there is a closed tableau for the latter, we can turn it into a closed tableau of the former by removing every formula that results from $\neg\mathbb{T}$ applied to the first assumption $\mathbb{T}\,\neg\varphi$ as well as that assumption, and adding the assumption $\mathbb{F}\,\varphi$. For if a branch was closed before because it contained the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{T}\,\neg\varphi$, i.e., $\mathbb{F}\,\varphi$, the corresponding branch in the new tableau is also closed. If a branch in the old tableau was closed because it contained the assumption $\mathbb{T}\,\neg\varphi$ as well as $\mathbb{F}\,\neg\varphi$ we can turn it into a closed branch by applying $\neg\mathbb{F}$ to $\mathbb{F}\,\neg\varphi$ to obtain $\mathbb{T}\,\varphi$. This closes the branch since we added $\mathbb{F}\,\varphi$ as an assumption. □

Problem tab.1. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:tab:prv: prop:explicit-inc

Proposition tab.3. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ such that

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

has a closed tableau. Replace the assumption $\mathbb{F}\,\varphi$ by $\mathbb{T}\,\neg\varphi$, and insert the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{F}\,\varphi$ after the assumptions. Any sentence in the tableau justified by appeal to line 1 in the old tableau is now justified by appeal to line $n + 1$. So if the old tableau was closed, the new one is. It shows that Γ is inconsistent, since all assumptions are in Γ. □

Proposition tab.4. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.* *fol:tab:prv: prop:provability-exhaustive*

Proof. If there are $\psi_1, \dots, \psi_n \in \Gamma$ and $\chi_1, \dots, \chi_m \in \Gamma$ such that

$$\{\mathbb{T}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$
$$\{\mathbb{T}\,\neg\varphi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_m\}$$

both have closed tableaux, we can construct a tableau that shows that Γ is inconsistent by using as assumptions $\mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n$ together with $\mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_m$, followed by an application of the Cut rule, yielding two branches, one starting with $\mathbb{T}\,\varphi$, the other with $\mathbb{F}\,\varphi$. Add on the part below the assumptions of the first tableau on the left side. Here, every rule application is still correct, and every branch closes. On the right side, add the part below the assumptions of the seond tableau, with the results of any applications of $\neg\mathbb{T}$ to $\mathbb{T}\,\neg\varphi$ removed.

For if a branch was closed before because it contained the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{T}\,\neg\varphi$, i.e., $\mathbb{F}\,\varphi$, as well as $\mathbb{F}\,\varphi$, the corresponding branch in the new tableau is also closed. If a branch in the old tableau was closed because it contained the assumption $\mathbb{T}\,\neg\varphi$ as well as $\mathbb{F}\,\neg\varphi$ we can turn it into a closed branch by applying $\neg\mathbb{F}$ to $\mathbb{F}\,\neg\varphi$ to obtain $\mathbb{T}\,\varphi$. □

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Bibliography