tab.1 Derivability and Consistency

fol:tab:prv: sec

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:tab:prv: prop:provability-contr

Proposition tab.1. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Proof. There are finite $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ and $\Gamma_1 = \{\chi_1, \dots, \chi_n\} \subseteq \Gamma$ such that

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$
$$\{\mathbb{T}\,\varphi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_m\}$$

have closed tableaux. Using the Cut rule on φ we can combine these into a single closed tableau that shows $\Gamma_0 \cup \Gamma_1$ is inconsistent. Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence Γ is inconsistent. □

fol:tab:prv: prop:prov-incons

Proposition tab.2. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a closed tableau for

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

Using the $\neg\mathbb{T}$ rule, this can be turned into a closed tableau for

$$\{\mathbb{T}\,\neg\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

On the other hand, if there is a closed tableau for the latter, we can turn it into a closed tableau of the former by removing every formula that results from $\neg\mathbb{T}$ applied to the first assumption $\mathbb{T}\,\neg\varphi$ as well as that assumption, and adding the assumption $\mathbb{F}\,\varphi$. For if a branch was closed before because it contained the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{T}\,\neg\varphi$, i.e., $\mathbb{F}\,\varphi$, the corresponding branch in the new tableau is also closed. If a branch in the old tableau was closed because it contained the assumption $\mathbb{T}\,\neg\varphi$ as well as $\mathbb{F}\,\neg\varphi$ we can turn it into a closed branch by applying $\neg\mathbb{F}$ to $\mathbb{F}\,\neg\varphi$ to obtain $\mathbb{T}\,\varphi$. This closes the branch since we added $\mathbb{F}\,\varphi$ as an assumption. □

Problem tab.1. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:tab:prv: prop:explicit-inc

Proposition tab.3. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ such that

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

has a closed tableau. Replace the assumption $\mathbb{F}\,\varphi$ by $\mathbb{T}\,\neg\varphi$, and insert the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{F}\,\varphi$ after the assumptions. Any sentence in the tableau justified by appeal to line 1 in the old tableau is now justified by appeal to line $n + 1$. So if the old tableau was closed, the new one is. It shows that Γ is inconsistent, since all assumptions are in Γ. □

Proposition tab.4. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.* fol:tab:prv: prop:provability-exhaustive

Proof. If there are $\psi_1, \ldots, \psi_n \in \Gamma$ and $\chi_1, \ldots, \chi_m \in \Gamma$ such that

$$\{\mathbb{T}\,\varphi, \mathbb{T}\,\psi_1, \ldots, \mathbb{T}\,\psi_n\} \text{ and}$$
$$\{\mathbb{T}\,\neg\varphi, \mathbb{T}\,\chi_1, \ldots, \mathbb{T}\,\chi_m\}$$

both have closed tableaux, we can construct a single, combined tableau that shows that Γ is inconsistent by using as assumptions $\mathbb{T}\,\psi_1, \ldots, \mathbb{T}\,\psi_n$ together with $\mathbb{T}\,\chi_1, \ldots, \mathbb{T}\,\chi_m$, followed by an application of the Cut rule. This yields two branches, one starting with $\mathbb{T}\,\varphi$, the other with $\mathbb{F}\,\varphi$.

On the left left side, add the part of the first tableau below its assumptions. Here, every rule application is still correct, since each of the assumptions of the first tableau, including $\mathbb{T}\,\varphi$, is available. Thus, every branch below $\mathbb{T}\,\varphi$ closes.

On the right side, add the part of the second tableau below its assumption, with the results of any applications of $\neg\mathbb{T}$ to $\mathbb{T}\,\neg\varphi$ removed. The conclusion of $\neg\mathbb{T}$ to $\mathbb{T}\,\neg\varphi$ is $\mathbb{F}\,\varphi$, which is nevertheless available, as it is the conclusion of the Cut rule on the right side of the combined tableau.

If a branch in the second tableau was closed because it contained the assumption $\mathbb{T}\,\neg\varphi$ (which no longer appears as an assumption in the combined tableau) as well as $\mathbb{F}\,\neg\varphi$, we can applying $\neg\mathbb{F}$ to $\mathbb{F}\,\neg\varphi$ to obtain $\mathbb{T}\,\varphi$. Now the corresponding branch in the combined tableau also closes, because it contains the right-hand conclusion of the Cut rule, $\mathbb{F}\,\varphi$. If a branch in the second tableau closed for any other reason, the corresponding branch in the combined tableau also closes, since any signed formulas other than $\mathbb{T}\,\neg\varphi$ occurring on the branch in the old, second tableau also occur on the corresponding branch in the combined tableau. □

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Bibliography