

tab.1 Proof-Theoretic Notions

fol:tab:ptn:
sec

This section collects the definitions of the provability relation and consistency for tableaux.

Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the existence of certain closed **tableaux**. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*. explanation

Definition tab.1 (Theorems). A **sentence** φ is a *theorem* if there is a closed **tableau** for $\mathbb{F}\varphi$. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition tab.2 (Derivability). A **sentence** φ is *derivable* from a set of **sentences** Γ , $\Gamma \vdash \varphi$, iff there is a finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ and a closed **tableau** for the set

$$\{\mathbb{F}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n, \}$$

If φ is not **derivable** from Γ we write $\Gamma \not\vdash \varphi$.

Definition tab.3 (Consistency). A set of **sentences** Γ is *inconsistent* iff there is a finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ and a closed **tableau** for the set

$$\{\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n, \}.$$

If Γ is not inconsistent, we say it is *consistent*.

fol:tab:ptn:
prop:reflexivity **Proposition tab.4** (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. If $\varphi \in \Gamma$, $\{\varphi\}$ is a finite subset of Γ and the **tableau**

1. $\mathbb{F}\varphi$ Assumption
 2. $\mathbb{T}\varphi$ Assumption
- ⊗

is closed. □

fol:tab:ptn:
prop:monotony **Proposition tab.5** (Monotony). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any finite subset of Γ is also a finite subset of Δ . □

fol:tab:ptn:
prop:transitivity **Proposition tab.6** (Transitivity). *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

Proof. If $\{\varphi\} \cup \Delta \vdash \psi$, then there is a finite subset $\Delta_0 = \{\chi_1, \dots, \chi_n\} \subseteq \Delta$ such that

$$\{\mathbb{F} \psi, \mathbb{T} \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n\}$$

has a closed **tableau**. If $\Gamma \vdash \varphi$ then there are $\theta_1, \dots, \theta_m$ such that

$$\{\mathbb{F} \varphi, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m\}$$

has a closed **tableau**.

Now consider the **tableau** with assumptions

$$\mathbb{F} \psi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m.$$

Apply the Cut rule on φ . This generates two branches, one has $\mathbb{T} \varphi$ in it, the other $\mathbb{F} \varphi$. Thus, on the one branch, all of

$$\{\mathbb{F} \psi, \mathbb{T} \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n\}$$

are available. Since there is a closed **tableau** for these assumptions, we can attach it to that branch; every branch through $\mathbb{T} \varphi_1$ closes. On the other branch, all of

$$\{\mathbb{F} \varphi, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m\}$$

are available, so we can also complete the other side to obtain a closed **tableau**. This shows $\Gamma \cup \Delta \vdash \psi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i , then $\Gamma \vdash \psi$.

Proposition tab.7. Γ is inconsistent iff $\Gamma \vdash \varphi$ for every **sentence** φ .

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prop:incons*

Proof. Exercise. □

Problem tab.1. Prove [Proposition tab.7](#)

Proposition tab.8 (Compactness).

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prop:proves-compact*

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of Γ is consistent, then Γ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ and a closed **tableau** for

$$\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n$$

This **tableau** also shows $\Gamma_0 \vdash \varphi$.

2. If Γ is inconsistent, then for some finite subset $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ there is a closed **tableau** for

$$\mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n$$

This closed **tableau** shows that Γ_0 is inconsistent.

□

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Bibliography