

## tab.1 Proof-Theoretic Notions

fol:tab:ptn:  
sec

This section collects the definitions of the provability relation and consistency for tableaux.

Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the existence of certain closed tableaux. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*.

explanation

**Definition tab.1** (Theorems). A sentence  $\varphi$  is a *theorem* if there is a closed tableau for  $\mathbb{F} \varphi$ . We write  $\vdash \varphi$  if  $\varphi$  is a theorem and  $\not\vdash \varphi$  if it is not.

**Definition tab.2** (Derivability). A sentence  $\varphi$  is *derivable* from a set of sentences  $\Gamma$ ,  $\Gamma \vdash \varphi$ , iff there is a finite set  $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$  and a closed tableau for the set

$$\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n, \}$$

If  $\varphi$  is not derivable from  $\Gamma$  we write  $\Gamma \not\vdash \varphi$ .

**Definition tab.3** (Consistency). A set of sentences  $\Gamma$  is *inconsistent* iff there is a finite set  $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$  and a closed tableau for the set

$$\{\mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n, \}.$$

If  $\Gamma$  is not inconsistent, we say it is *consistent*.

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prop:reflexivity

**Proposition tab.4** (Reflexivity). If  $\varphi \in \Gamma$ , then  $\Gamma \vdash \varphi$ .

*Proof.* If  $\varphi \in \Gamma$ ,  $\{\varphi\}$  is a finite subset of  $\Gamma$  and the tableau

1.  $\mathbb{F} \varphi$  Assumption
  2.  $\mathbb{T} \varphi$  Assumption
- ⊗

is closed. □

fol:tab:ptn:  
prop:monotony

**Proposition tab.5** (Monotony). If  $\Gamma \subseteq \Delta$  and  $\Gamma \vdash \varphi$ , then  $\Delta \vdash \varphi$ .

*Proof.* Any finite subset of  $\Gamma$  is also a finite subset of  $\Delta$ . □

fol:tab:ptn:  
prop:transitivity

**Proposition tab.6** (Transitivity). If  $\Gamma \vdash \varphi$  and  $\{\varphi\} \cup \Delta \vdash \psi$ , then  $\Gamma \cup \Delta \vdash \psi$ .

*Proof.* If  $\{\varphi\} \cup \Delta \vdash \psi$ , then there is a finite subset  $\Delta_0 = \{\chi_1, \dots, \chi_n\} \subseteq \Delta$  such that

$$\{\mathbb{F} \psi, \mathbb{T} \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n\}$$

has a closed **tableau**. If  $\Gamma \vdash \varphi$  then there are  $\theta_1, \dots, \theta_m$  such that

$$\{\mathbb{F} \varphi, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m\}$$

has a closed **tableau**.

Now consider the **tableau** with assumptions

$$\mathbb{F} \psi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m.$$

Apply the Cut rule on  $\varphi$ . This generates two branches, one has  $\mathbb{T} \varphi$  in it, the other  $\mathbb{F} \varphi$ . Thus, on the one branch, all of

$$\{\mathbb{F} \psi, \mathbb{T} \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n\}$$

are available. Since there is a closed **tableau** for these assumptions, we can attach it to that branch; every branch through  $\mathbb{T} \varphi_1$  closes. On the other branch, all of

$$\{\mathbb{F} \varphi, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m\}$$

are available, so we can also complete the other side to obtain a closed **tableau**. This shows  $\Gamma \cup \Delta \vdash \psi$ . □

Note that this means that in particular if  $\Gamma \vdash \varphi$  and  $\varphi \vdash \psi$ , then  $\Gamma \vdash \psi$ . It follows also that if  $\varphi_1, \dots, \varphi_n \vdash \psi$  and  $\Gamma \vdash \varphi_i$  for each  $i$ , then  $\Gamma \vdash \psi$ .

**Proposition tab.7.**  $\Gamma$  is inconsistent iff  $\Gamma \vdash \varphi$  for every **sentence**  $\varphi$ .

*fol:tab:ptn:  
prop:incons*

*Proof.* Exercise. □

**Problem tab.1.** Prove [Proposition tab.7](#)

**Proposition tab.8** (Compactness).

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prop:proves-compact*

1. If  $\Gamma \vdash \varphi$  then there is a finite subset  $\Gamma_0 \subseteq \Gamma$  such that  $\Gamma_0 \vdash \varphi$ .
2. If every finite subset of  $\Gamma$  is consistent, then  $\Gamma$  is consistent.

*Proof.* 1. If  $\Gamma \vdash \varphi$ , then there is a finite subset  $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$  and a closed **tableau** for

$$\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n$$

This **tableau** also shows  $\Gamma_0 \vdash \varphi$ .

2. If  $\Gamma$  is inconsistent, then for some finite subset  $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$  there is a closed **tableau** for

$$\mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n$$

This closed **tableau** shows that  $\Gamma_0$  is inconsistent.

□

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## Bibliography