tab.1 Tableaux with Identity predicate

fol:tab:ide: sec Tableaux with identity predicate require additional inference rules. The rules for $=$ are (t, t_1, and t_2 are closed terms):

$$\frac{}{\mathbb{T}\, t = t} = \qquad \frac{\begin{array}{c}\mathbb{T}\, t_1 = t_2 \\ \mathbb{T}\, \varphi(t_1)\end{array}}{\mathbb{T}\, \varphi(t_1)} =\mathbb{T} \qquad \frac{\begin{array}{c}\mathbb{T}\, t_1 = t_2 \\ \mathbb{F}\, \varphi(t_1)\end{array}}{\mathbb{F}\, \varphi(t_1)} =\mathbb{T}$$

Note that in contrast to all the other rules, $=\mathbb{T}$ and $=\mathbb{F}$ require that *two* signed formulas already appear on the branch, namely both $\mathbb{T}\, t_1 = t_2$ and $S\, \varphi(t_1)$.

Example tab.1. If s and t are closed terms, then $s = t, \varphi(s) \vdash \varphi(t)$:

1.	$\mathbb{F}\, \varphi(t)$	Assumption
2.	$\mathbb{T}\, s = t$	Assumption
3.	$\mathbb{T}\, \varphi(s)$	Assumption
4.	$\mathbb{T}\, \varphi(t)$	$=\mathbb{T}\, 2, 3$
	$\otimes$	

This may be familiar as the principle of substitutability of identicals, or Leibniz' Law.

Tableaux prove that $=$ is symmetric:

1.	$\mathbb{F}\, t = s$	Assumption
2.	$\mathbb{T}\, s = t$	Assumption
3.	$\mathbb{T}\, s = s$	$=$
4.	$\mathbb{T}\, t = s$	$=\mathbb{T}\, 2, 3$
	$\otimes$	

Here, line 2 is the first prerequisite formula $\mathbb{T}\, s = t$ of $=\mathbb{T}$, and line 3 the second one, $\mathbb{T}\, \varphi(s)$—think of $\varphi(x)$ as $x = s$, then $\varphi(s)$ is $s = s$ and $\varphi(t)$ is $t = s$.

They also prove that $=$ is transitive:

1.	$\mathbb{F}\, t_1 = t_3$	Assumption
2.	$\mathbb{T}\, t_1 = t_2$	Assumption
3.	$\mathbb{T}\, t_2 = t_3$	Assumption
4.	$\mathbb{T}\, t_1 = t_3$	$=\mathbb{T}\, 3, 2$
	$\otimes$	

In this tableau, the first prerequisite formula of $=\mathbb{T}$ is line 3, $\mathbb{T}\, t_2 = t_3$. The second one, $\mathbb{T}\, \varphi(t_2)$ is line 2. Think of $\varphi(x)$ as $t_1 = x$; that makes $\varphi(t_2)$ into $t_1 = t_2$ and $\varphi(t_3)$ into $t_1 = t_3$.

Problem tab.1. Give closed tableaux for the following:

1. $\mathbb{F}\, \forall x\, \forall y\, ((x = y \land \varphi(x)) \to \varphi(y))$
2. $\mathbb{F}\, \exists x\, (\varphi(x) \land \forall y\, (\varphi(y) \to y = x))$,
   $\mathbb{T}\, \exists x\, \varphi(x) \land \forall y\, \forall z\, ((\varphi(y) \land \varphi(z)) \to y = z)$

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Bibliography