tab.1 Tableaux with Identity predicate

Tableaux with identity predicate require additional inference rules. The rules for $=$ are (t, t_1, and t_2 are closed terms):

$$\overline{\mathbb{T}\,t = t}\,=\qquad\qquad \frac{\begin{array}{c}\mathbb{T}\,t_1 = t_2\\\mathbb{T}\,\varphi(t_1)\end{array}}{\mathbb{T}\,\varphi(t_1)}{=}\mathbb{T}\qquad\qquad \frac{\begin{array}{c}\mathbb{T}\,t_1 = t_2\\\mathbb{F}\,\varphi(t_1)\end{array}}{\mathbb{F}\,\varphi(t_1)}{=}\mathbb{T}$$

Note that in contrast to all the other rules, $=\mathbb{T}$ and $=\mathbb{F}$ require that *two* signed formulas already appear on the branch, namely both $\mathbb{T}\,t_1 = t_2$ and $S\,\varphi(t_1)$.

Example tab.1. If s and t are closed terms, then $s = t, \varphi(s) \vdash \varphi(t)$:

$$\begin{array}{llll}
1. & \mathbb{F}\,\varphi(t) & \text{Assumption} \\
2. & \mathbb{T}\,s = t & \text{Assumption} \\
3. & \mathbb{T}\,\varphi(s) & \text{Assumption} \\
4. & \mathbb{T}\,\varphi(t) & =\mathbb{T}\,2, 3 \\
& \otimes
\end{array}$$

This may be familiar as the principle of substitutability of identicals, or Leibniz' Law.

Tableaux prove that $=$ is symmetric:

$$\begin{array}{llll}
1. & \mathbb{F}\,t = s & \text{Assumption} \\
2. & \mathbb{T}\,s = t & \text{Assumption} \\
3. & \mathbb{T}\,s = s & = \\
4. & \mathbb{T}\,t = s & =\mathbb{T}\,2, 3 \\
& \otimes
\end{array}$$

Here, line 2 is the first prerequisite formula $\mathbb{T}\,s = t$ of $=\mathbb{T}$, and line 3 the second one, $\mathbb{T}\,\varphi(s)$—think of $\varphi(x)$ as $x = s$, then $\varphi(s)$ is $s = s$ and $\varphi(t)$ is $t = s$.

They also prove that $=$ is transitive:

$$\begin{array}{llll}
1. & \mathbb{F}\,t_1 = t_3 & \text{Assumption} \\
2. & \mathbb{T}\,t_1 = t_2 & \text{Assumption} \\
3. & \mathbb{T}\,t_2 = t_3 & \text{Assumption} \\
4. & \mathbb{T}\,t_1 = t_3 & =\mathbb{T}\,3, 2 \\
& \otimes
\end{array}$$

In this tableau, the first prerequisite formula of $=\mathbb{T}$ is line 3, $\mathbb{T}\,t_2 = t_3$. The second one, $\mathbb{T}\,\varphi(t_2)$ is line 2. Think of $\varphi(x)$ as $t_1 = x$; that makes $\varphi(t_2)$ into $t_1 = t_2$ and $\varphi(t_3)$ into $t_1 = t_3$.

Problem tab.1. Give closed tableaux for the following:

1. $\mathbb{F} \, \forall x \, \forall y \, ((x = y \wedge \varphi(x)) \rightarrow \varphi(y))$

2. $\mathbb{F} \, \exists x \, (\varphi(x) \wedge \forall y \, (\varphi(y) \rightarrow y = x))$,
   $\mathbb{T} \, \exists x \, \varphi(x) \wedge \forall y \, \forall z \, ((\varphi(y) \wedge \varphi(z)) \rightarrow y = z)$

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Bibliography