syn.1 Subformulas

fol:syn:sbf:sec It is often useful to talk about the formulas that “make up” a given formula. explanation
We call these its *subformulas*. Any formula counts as a subformula of itself; a subformula of φ other than φ itself is a *proper subformula*.

Definition syn.1 (Immediate Subformula). If φ is a formula, the *immediate subformulas* of φ are defined inductively as follows:

1. Atomic formulas have no immediate subformulas.
2. $\varphi \equiv \lnot\psi$: The only immediate subformula of φ is ψ.
3. $\varphi \equiv (\psi * \chi)$: The immediate subformulas of φ are ψ and χ ($*$ is any one of the two-place connectives).
4. $\varphi \equiv \forall x\, \psi$: The only immediate subformula of φ is ψ.
5. $\varphi \equiv \exists x\, \psi$: The only immediate subformula of φ is ψ.

Definition syn.2 (Proper Subformula). If φ is a formula, the *proper subformulas* of φ are defined recursively as follows:

1. Atomic formulas have no proper subformulas.
2. $\varphi \equiv \lnot\psi$: The proper subformulas of φ are ψ together with all proper subformulas of ψ.
3. $\varphi \equiv (\psi * \chi)$: The proper subformulas of φ are ψ, χ, together with all proper subformulas of ψ and those of χ.
4. $\varphi \equiv \forall x\, \psi$: The proper subformulas of φ are ψ together with all proper subformulas of ψ.
5. $\varphi \equiv \exists x\, \psi$: The proper subformulas of φ are ψ together with all proper subformulas of ψ.

Definition syn.3 (Subformula). The subformulas of φ are φ itself together with all its proper subformulas.

Note the subtle difference in how we have defined immediate subformulas explanation
and proper subformulas. In the first case, we have directly defined the immediate subformulas of a formula φ for each possible form of φ. It is an explicit definition by cases, and the cases mirror the inductive definition of the set of formulas. In the second case, we have also mirrored the way the set of all formulas is defined, but in each case we have also included the proper subformulas of the smaller formulas ψ, χ in addition to these formulas themselves. This makes the definition *recursive*. In general, a definition of a function on an inductively defined set (in our case, formulas) is recursive if the cases in the definition of the function make use of the function itself. To be well defined,

we must make sure, however, that we only ever use the values of the function for arguments that come "before" the one we are defining—in our case, when defining "proper subformula" for $(\psi * \chi)$ we only use the proper subformulas of the "earlier" formulas ψ and χ.

Proposition syn.4. *Suppose ψ is a subformula of φ and χ is a subformula of ψ. Then χ is a subformula of φ. In other words, the subformula relation is transitive.* *fol:syn:sbf:prop:subfrm-trans*

Problem syn.1. Prove Proposition syn.4.

Proposition syn.5. *Suppose φ is a formula with n connectives and quantifiers. Then φ has at most $2n + 1$ subformulas.* *fol:syn:sbf:prop:count-subfrms*

Problem syn.2. Prove Proposition syn.5.

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Bibliography