

syn.1 Structures for First-order Languages

fol:syn:str:
sec

First-order languages are, by themselves, *uninterpreted*: the **constant symbols**, **function symbols**, and **predicate symbols** have no specific meaning attached to them. Meanings are given by specifying a *structure*. It specifies the *domain*, i.e., the objects which the **constant symbols** pick out, the **function symbols** operate on, and the quantifiers range over. In addition, it specifies which **constant symbols** pick out which objects, how a **function symbol** maps objects to objects, and which objects the **predicate symbols** apply to. **Structures** are the basis for *semantic* notions in logic, e.g., the notion of consequence, validity, satisfiability. They are variously called “structures,” “interpretations,” or “models” in the literature.

explanation

Definition syn.1 (Structures). A *structure* \mathfrak{M} , for a language \mathcal{L} of first-order logic consists of the following elements:

1. *Domain*: a non-empty set, $|\mathfrak{M}|$
2. *Interpretation of constant symbols*: for each **constant symbol** c of \mathcal{L} , an **element** $c^{\mathfrak{M}} \in |\mathfrak{M}|$
3. *Interpretation of predicate symbols*: for each n -place **predicate symbol** R of \mathcal{L} (other than $=$), an n -place relation $R^{\mathfrak{M}} \subseteq |\mathfrak{M}|^n$
4. *Interpretation of function symbols*: for each n -place **function symbol** f of \mathcal{L} , an n -place function $f^{\mathfrak{M}}: |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$

Example syn.2. A *structure* \mathfrak{M} for the language of arithmetic consists of a set, an element of $|\mathfrak{M}|$, $o^{\mathfrak{M}}$, as interpretation of the **constant symbol** o , a one-place function $\iota^{\mathfrak{M}}: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$, two two-place functions $+^{\mathfrak{M}}$ and $\times^{\mathfrak{M}}$, both $|\mathfrak{M}|^2 \rightarrow |\mathfrak{M}|$, and a two-place relation $<^{\mathfrak{M}} \subseteq |\mathfrak{M}|^2$.

An obvious example of such a structure is the following:

1. $|\mathfrak{N}| = \mathbb{N}$
2. $o^{\mathfrak{N}} = 0$
3. $\iota^{\mathfrak{N}}(n) = n + 1$ for all $n \in \mathbb{N}$
4. $+^{\mathfrak{N}}(n, m) = n + m$ for all $n, m \in \mathbb{N}$
5. $\times^{\mathfrak{N}}(n, m) = n \cdot m$ for all $n, m \in \mathbb{N}$
6. $<^{\mathfrak{N}} = \{(n, m) : n \in \mathbb{N}, m \in \mathbb{N}, n < m\}$

The structure \mathfrak{N} for \mathcal{L}_A so defined is called the *standard model of arithmetic*, because it interprets the non-logical constants of \mathcal{L}_A exactly how you would expect.

However, there are many other possible **structures** for \mathcal{L}_A . For instance, we might take as the domain the set \mathbb{Z} of integers instead of \mathbb{N} , and define the interpretations of o , ι , $+$, \times , $<$ accordingly. But we can also define structures for \mathcal{L}_A which have nothing even remotely to do with numbers.

Example syn.3. A structure \mathfrak{M} for the language \mathcal{L}_Z of set theory requires just a set and a single-two place relation. So technically, e.g., the set of people plus the relation “ x is older than y ” could be used as a **structure** for \mathcal{L}_Z , as well as \mathbb{N} together with $n \geq m$ for $n, m \in \mathbb{N}$.

A particularly interesting **structure** for \mathcal{L}_Z in which the **elements** of the domain are actually sets, and the interpretation of \in actually is the relation “ x is an **element** of y ” is the **structure** $\mathfrak{H}\mathfrak{F}$ of *hereditarily finite sets*:

1. $|\mathfrak{H}\mathfrak{F}| = \emptyset \cup \wp(\emptyset) \cup \wp(\wp(\emptyset)) \cup \wp(\wp(\wp(\emptyset))) \cup \dots;$
2. $\in^{\mathfrak{H}\mathfrak{F}} = \{\langle x, y \rangle : x, y \in |\mathfrak{H}\mathfrak{F}|, x \in y\}.$

digression

The stipulations we make as to what counts as a **structure** impact our logic. For example, the choice to prevent empty domains ensures, given the usual account of satisfaction (or truth) for quantified sentences, that $\exists x (\varphi(x) \vee \neg\varphi(x))$ is valid—that is, a logical truth. And the stipulation that all **constant symbols** must refer to an object in the domain ensures that the existential generalization is a sound pattern of inference: $\varphi(a)$, therefore $\exists x \varphi(x)$. If we allowed names to refer outside the domain, or to not refer, then we would be on our way to a *free logic*, in which existential generalization requires an additional premise: $\varphi(a)$ and $\exists x x = a$, therefore $\exists x \varphi(x)$.

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Bibliography