

## syn.1 Structures for First-order Languages

fol:syn:str:  
sec

First-order languages are, by themselves, *uninterpreted*: the **constant symbols**, **function symbols**, and **predicate symbols** have no specific meaning attached to them. Meanings are given by specifying a *structure*. It specifies the *domain*, i.e., the objects which the **constant symbols** pick out, the **function symbols** operate on, and the quantifiers range over. In addition, it specifies which **constant symbols** pick out which objects, how a **function symbol** maps objects to objects, and which objects the **predicate symbols** apply to. **Structures** are the basis for *semantic* notions in logic, e.g., the notion of consequence, validity, satisfiability. They are variously called “structures,” “interpretations,” or “models” in the literature.

explanation

**Definition syn.1 (Structures).** A *structure*  $\mathfrak{M}$ , for a language  $\mathcal{L}$  of first-order logic consists of the following elements:

1. *Domain*: a non-empty set,  $|\mathfrak{M}|$
2. *Interpretation of constant symbols*: for each **constant symbol**  $c$  of  $\mathcal{L}$ , an **element**  $c^{\mathfrak{M}} \in |\mathfrak{M}|$
3. *Interpretation of predicate symbols*: for each  $n$ -place **predicate symbol**  $R$  of  $\mathcal{L}$  (other than  $=$ ), an  $n$ -place relation  $R^{\mathfrak{M}} \subseteq |\mathfrak{M}|^n$
4. *Interpretation of function symbols*: for each  $n$ -place **function symbol**  $f$  of  $\mathcal{L}$ , an  $n$ -place function  $f^{\mathfrak{M}}: |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$

**Example syn.2.** A *structure*  $\mathfrak{M}$  for the language of arithmetic consists of a set, an element of  $|\mathfrak{M}|$ ,  $o^{\mathfrak{M}}$ , as interpretation of the **constant symbol**  $o$ , a one-place function  $\iota^{\mathfrak{M}}: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$ , two two-place functions  $+^{\mathfrak{M}}$  and  $\times^{\mathfrak{M}}$ , both  $|\mathfrak{M}|^2 \rightarrow |\mathfrak{M}|$ , and a two-place relation  $<^{\mathfrak{M}} \subseteq |\mathfrak{M}|^2$ .

An obvious example of such a structure is the following:

1.  $|\mathfrak{N}| = \mathbb{N}$
2.  $o^{\mathfrak{N}} = 0$
3.  $\iota^{\mathfrak{N}}(n) = n + 1$  for all  $n \in \mathbb{N}$
4.  $+^{\mathfrak{N}}(n, m) = n + m$  for all  $n, m \in \mathbb{N}$
5.  $\times^{\mathfrak{N}}(n, m) = n \cdot m$  for all  $n, m \in \mathbb{N}$
6.  $<^{\mathfrak{N}} = \{(n, m) : n \in \mathbb{N}, m \in \mathbb{N}, n < m\}$

The structure  $\mathfrak{N}$  for  $\mathcal{L}_A$  so defined is called the *standard model of arithmetic*, because it interprets the non-logical constants of  $\mathcal{L}_A$  exactly how you would expect.

However, there are many other possible **structures** for  $\mathcal{L}_A$ . For instance, we might take as the domain the set  $\mathbb{Z}$  of integers instead of  $\mathbb{N}$ , and define the interpretations of  $o$ ,  $\iota$ ,  $+$ ,  $\times$ ,  $<$  accordingly. But we can also define structures for  $\mathcal{L}_A$  which have nothing even remotely to do with numbers.

**Example syn.3.** A structure  $\mathfrak{M}$  for the language  $\mathcal{L}_Z$  of set theory requires just a set and a single-two place relation. So technically, e.g., the set of people plus the relation “ $x$  is older than  $y$ ” could be used as a **structure** for  $\mathcal{L}_Z$ , as well as  $\mathbb{N}$  together with  $n \geq m$  for  $n, m \in \mathbb{N}$ .

A particularly interesting **structure** for  $\mathcal{L}_Z$  in which the **elements** of the domain are actually sets, and the interpretation of  $\in$  actually is the relation “ $x$  is an **element** of  $y$ ” is the **structure**  $\mathfrak{H}\mathfrak{F}$  of *hereditarily finite sets*:

1.  $|\mathfrak{H}\mathfrak{F}| = \emptyset \cup \wp(\emptyset) \cup \wp(\wp(\emptyset)) \cup \wp(\wp(\wp(\emptyset))) \cup \dots;$
2.  $\in^{\mathfrak{H}\mathfrak{F}} = \{\langle x, y \rangle : x, y \in |\mathfrak{H}\mathfrak{F}|, x \in y\}.$

**digression**

The stipulations we make as to what counts as a **structure** impact our logic. For example, the choice to prevent empty domains ensures, given the usual account of satisfaction (or truth) for quantified sentences, that  $\exists x (\varphi(x) \vee \neg\varphi(x))$  is valid—that is, a logical truth. And the stipulation that all **constant symbols** must refer to an object in the domain ensures that the existential generalization is a sound pattern of inference:  $\varphi(a)$ , therefore  $\exists x \varphi(x)$ . If we allowed names to refer outside the domain, or to not refer, then we would be on our way to a *free logic*, in which existential generalization requires an additional premise:  $\varphi(a)$  and  $\exists x x = a$ , therefore  $\exists x \varphi(x)$ .

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## Bibliography