

syn.1 Semantic Notions

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sec

Give the definition of **structures** for first-order languages, we can define **explanation** some basic semantic properties of and relationships between sentences. The simplest of these is the notion of *validity* of a sentence. A sentence is valid if it is satisfied in every **structure**. Valid sentences are those that are satisfied regardless of how the non-logical symbols in it are interpreted. Valid sentences are therefore also called *logical truths*—they are true, i.e., satisfied, in any **structure** and hence their truth depends only on the logical symbols occurring in them and their syntactic **structure**, but not on the non-logical symbols or their interpretation.

Definition syn.1 (Validity). A sentence φ is *valid*, $\models \varphi$, iff $\mathfrak{M} \models \varphi$ for every **structure** \mathfrak{M} .

Definition syn.2 (Entailment). A set of sentences Γ *entails* a sentence φ , $\Gamma \models \varphi$, iff for every **structure** \mathfrak{M} with $\mathfrak{M} \models \Gamma$, $\mathfrak{M} \models \varphi$.

Definition syn.3 (Satisfiability). A set of sentences Γ is *satisfiable* if $\mathfrak{M} \models \Gamma$ for some **structure** \mathfrak{M} . If Γ is not satisfiable it is called *unsatisfiable*.

Proposition syn.4. A sentence φ is valid iff $\Gamma \models \varphi$ for every set of sentences Γ .

Proof. For the forward direction, let φ be valid, and let Γ be a set of sentences. Let \mathfrak{M} be a **structure** so that $\mathfrak{M} \models \Gamma$. Since φ is valid, $\mathfrak{M} \models \varphi$, hence $\Gamma \models \varphi$.

For the contrapositive of the reverse direction, let φ be invalid, so there is a **structure** \mathfrak{M} with $\mathfrak{M} \not\models \varphi$. When $\Gamma = \{\top\}$, since \top is valid, $\mathfrak{M} \models \Gamma$. Hence, there is a **structure** \mathfrak{M} so that $\mathfrak{M} \models \Gamma$ but $\mathfrak{M} \not\models \varphi$, hence Γ does not entail φ . \square

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prop:entails-unsat

Proposition syn.5. $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable.

Proof. For the forward direction, suppose $\Gamma \models \varphi$ and suppose to the contrary that there is a **structure** \mathfrak{M} so that $\mathfrak{M} \models \Gamma \cup \{\neg\varphi\}$. Since $\mathfrak{M} \models \Gamma$ and $\Gamma \models \varphi$, $\mathfrak{M} \models \varphi$. Also, since $\mathfrak{M} \models \Gamma \cup \{\neg\varphi\}$, $\mathfrak{M} \models \neg\varphi$, so we have both $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \models \neg\varphi$, a contradiction. Hence, there can be no such **structure** \mathfrak{M} , so $\Gamma \cup \{\varphi\}$ is unsatisfiable.

For the reverse direction, suppose $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable. So for every **structure** \mathfrak{M} , either $\mathfrak{M} \not\models \Gamma$ or $\mathfrak{M} \models \varphi$. Hence, for every **structure** \mathfrak{M} with $\mathfrak{M} \models \Gamma$, $\mathfrak{M} \models \varphi$, so $\Gamma \models \varphi$. \square

Problem syn.1. 1. Show that $\Gamma \models \perp$ iff Γ is unsatisfiable.

2. Show that $\Gamma \cup \{\varphi\} \models \perp$ iff $\Gamma \models \neg\varphi$.

3. Suppose c does not occur in φ or Γ . Show that $\Gamma \models \forall x \varphi$ iff $\Gamma \models \varphi[c/x]$.

Proposition syn.6. If $\Gamma \subseteq \Gamma'$ and $\Gamma \models \varphi$, then $\Gamma' \models \varphi$.

Proof. Suppose that $\Gamma \subseteq \Gamma'$ and $\Gamma \models \varphi$. Let \mathfrak{M} be such that $\mathfrak{M} \models \Gamma'$; then $\mathfrak{M} \models \Gamma$, and since $\Gamma \models \varphi$, we get that $\mathfrak{M} \models \varphi$. Hence, whenever $\mathfrak{M} \models \Gamma'$, $\mathfrak{M} \models \varphi$, so $\Gamma' \models \varphi$. \square

Theorem syn.7 (Semantic Deduction Theorem). $\Gamma \cup \{\varphi\} \models \psi$ iff $\Gamma \models \varphi \rightarrow \psi$. fol:syn:sem: thm:sem-deduction

Proof. For the forward direction, let $\Gamma \cup \{\varphi\} \models \psi$ and let \mathfrak{M} be a **structure** so that $\mathfrak{M} \models \Gamma$. If $\mathfrak{M} \models \varphi$, then $\mathfrak{M} \models \Gamma \cup \{\varphi\}$, so since $\Gamma \cup \{\varphi\}$ entails ψ , we get $\mathfrak{M} \models \psi$. Therefore, $\mathfrak{M} \models \varphi \rightarrow \psi$, so $\Gamma \models \varphi \rightarrow \psi$.

For the reverse direction, let $\Gamma \models \varphi \rightarrow \psi$ and \mathfrak{M} be a **structure** so that $\mathfrak{M} \models \Gamma \cup \{\varphi\}$. Then $\mathfrak{M} \models \Gamma$, so $\mathfrak{M} \models \varphi \rightarrow \psi$, and since $\mathfrak{M} \models \varphi$, $\mathfrak{M} \models \psi$. Hence, whenever $\mathfrak{M} \models \Gamma \cup \{\varphi\}$, $\mathfrak{M} \models \psi$, so $\Gamma \cup \{\varphi\} \models \psi$. \square

Proposition syn.8. Let \mathfrak{M} be a **structure**, and $\varphi(x)$ a **formula** with one free variable x , and t a closed term. Then: fol:syn:sem: prop:quant-terms

1. $\varphi(t) \models \exists x \varphi(x)$
2. $\forall x \varphi(x) \models \varphi(t)$

Proof. 1. Suppose $\mathfrak{M} \models \varphi(t)$. Let s be a variable assignment with $s(x) = \text{Val}^{\mathfrak{M}}(t)$. Then $\mathfrak{M}, s \models \varphi(t)$ since $\varphi(t)$ is a **sentence**. By ??, $\mathfrak{M}, s \models \varphi(x)$. By ??, $\mathfrak{M} \models \exists x \varphi(x)$.

2. Suppose $\mathfrak{M} \models \forall x \varphi(x)$. Let s be a variable assignment with $s(x) = \text{Val}^{\mathfrak{M}}(t)$. By ??, $\mathfrak{M}, s \models \varphi(x)$. By ??, $\mathfrak{M}, s \models \varphi(t)$. By ??, $\mathfrak{M} \models \varphi(t)$ since $\varphi(t)$ is a **sentence**. \square

Problem syn.2. Complete the proof of **Proposition syn.8**.

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Bibliography