syn.1 Semantic Notions

Give the definition of structures for first-order languages, we can define some basic semantic properties of and relationships between sentences. The simplest of these is the notion of **validity** of a sentence. A sentence is valid if it is satisfied in every structure. Valid sentences are those that are satisfied regardless of how the non-logical symbols in it are interpreted. Valid sentences are therefore also called **logical truths**—they are true, i.e., satisfied, in any structure and hence their truth depends only on the logical symbols occurring in them and their syntactic structure, but not on the non-logical symbols or their interpretation.

**Definition syn.1 (Validity).** A sentence \( \varphi \) is **valid**, \( \vdash \varphi \), iff \( M \models \varphi \) for every structure \( M \).

**Definition syn.2 (Entailment).** A set of sentences \( \Gamma \) **entails** a sentence \( \varphi \), \( \Gamma \vdash \varphi \), iff for every structure \( M \) with \( M \models \Gamma \), \( M \models \varphi \).

**Definition syn.3 (Satisfiability).** A set of sentences \( \Gamma \) is **satisfiable** if \( M \models \Gamma \) for some structure \( M \). If \( \Gamma \) is not satisfiable it is called **unsatisfiable**.

**Proposition syn.4.** A sentence \( \varphi \) is valid iff \( \Gamma \models \varphi \) for every set of sentences \( \Gamma \).

*Proof.* For the forward direction, let \( \varphi \) be valid, and let \( \Gamma \) be a set of sentences. Let \( M \) be a structure so that \( M \models \Gamma \). Since \( \varphi \) is valid, \( M \models \varphi \), hence \( \Gamma \models \varphi \).

For the contrapositive of the reverse direction, let \( \varphi \) be invalid, so there is a structure \( M \) with \( M \not\models \varphi \). When \( \Gamma = \{\top\} \), since \( \top \) is valid, \( M \models \Gamma \). Hence, there is a structure \( M \) so that \( M \models \Gamma \) but \( M \not\models \varphi \), hence \( \Gamma \) does not entail \( \varphi \).

**Proposition syn.5.** \( \Gamma \models \varphi \) iff \( \Gamma \cup \{\neg \varphi\} \) is unsatisfiable.

*Proof.* For the forward direction, suppose \( \Gamma \models \varphi \) and suppose to the contrary that there is a structure \( M \) so that \( M \models \Gamma \cup \{\neg \varphi\} \). Since \( M \models \Gamma \) and \( \Gamma \models \varphi \), \( M \models \varphi \). Also, since \( M \models \Gamma \cup \{\neg \varphi\} \), \( M \models \neg \varphi \), so we have both \( M \models \varphi \) and \( M \not\models \varphi \), a contradiction. Hence, there can be no such structure \( M \), so \( \Gamma \cup \{\varphi\} \) is unsatisfiable.

For the reverse direction, suppose \( \Gamma \cup \{\neg \varphi\} \) is unsatisfiable. So for every structure \( M \), either \( M \not\models \Gamma \) or \( M \models \varphi \). Hence, for every structure \( M \) with \( M \models \Gamma \), \( M \models \varphi \), so \( \Gamma \models \varphi \).

**Problem syn.1.**
1. Show that \( \Gamma \models \bot \) iff \( \Gamma \) is unsatisfiable.
2. Show that \( \Gamma \cup \{\varphi\} \models \bot \) iff \( \Gamma \models \neg \varphi \).
3. Suppose \( c \) does not occur in \( \varphi \) or \( \Gamma \). Show that \( \Gamma \models \forall x \varphi \) iff \( \Gamma \models \varphi[c/x] \).

**Proposition syn.6.** If \( \Gamma \subseteq \Gamma' \) and \( \Gamma \models \varphi \), then \( \Gamma' \models \varphi \).
Proof. Suppose that $\Gamma \subseteq \Gamma'$ and $\Gamma \models \varphi$. Let $M$ be such that $M \models \Gamma'$; then $M \models \Gamma$, and since $\Gamma' \models \varphi$, we get that $M \models \varphi$. Hence, whenever $M \models \Gamma'$, $M \models \varphi$, so $\Gamma' \models \varphi$.

Theorem syn.7 (Semantic Deduction Theorem). $\Gamma \cup \{\varphi\} \models \psi$ iff $\Gamma \models \varphi \rightarrow \psi$.

Proof. For the forward direction, let $\Gamma \cup \{\varphi\} \models \psi$ and let $M$ be a structure so that $M \models \Gamma$. If $M \models \varphi$, then $M \models \Gamma \cup \{\varphi\}$, so since $\Gamma \cup \{\varphi\}$ entails $\psi$, we get $M \models \psi$. Therefore, $M \models \varphi \rightarrow \psi$, so $\Gamma \models \varphi \rightarrow \psi$.

For the reverse direction, let $\Gamma \models \varphi \rightarrow \psi$ and $M$ be a structure so that $M \models \Gamma \cup \{\varphi\}$. Then $M \models \Gamma$, so $M \models \varphi \rightarrow \psi$, and since $M \models \varphi$, $M \models \psi$. Hence, whenever $M \models \Gamma \cup \{\varphi\}$, $M \models \psi$, so $\Gamma \cup \{\varphi\} \models \psi$. 

Proposition syn.8. Let $M$ be a structure, and $\varphi(x)$ a formula with one free variable $x$, and $t$ a closed term. Then:

1. $\varphi(t) \models \exists x \varphi(x)$
2. $\forall x \varphi(x) \models \varphi(t)$

Proof. 1. Suppose $M \models \varphi(t)$. Let $s$ be a variable assignment with $s(x) = \text{Val}_M(t)$. Then $M, s \models \varphi(t)$ since $\varphi(t)$ is a sentence. By $??$, $M, s \models \varphi(x)$. By $??$, $M \models \exists x \varphi(x)$.

2. Suppose $M \models \forall x \varphi(x)$. Let $s$ be a variable assignment with $s(x) = \text{Val}_M(t)$. By $??$, $M, s \models \varphi(x)$. By $??$, $M, s \models \varphi(t)$. By $??$, $M \models \varphi(t)$ since $\varphi(t)$ is a sentence.

Problem syn.2. Complete the proof of Proposition syn.8.

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Bibliography