syn.1 Semantic Notions

fol:syn:sem:sec

Give the definition of structures for first-order languages, we can define some basic semantic properties of and relationships between sentences. The simplest of these is the notion of *validity* of a sentence. A sentence is valid if it is satisfied in every structure. Valid sentences are those that are satisfied regardless of how the non-logical symbols in it are interpreted. Valid sentences are therefore also called *logical truths*—they are true, i.e., satisfied, in any structure and hence their truth depends only on the logical symbols occurring in them and their syntactic structure, but not on the non-logical symbols or their interpretation. explanation

Definition syn.1 (Validity)**.** A sentence φ is *valid*, $\vDash \varphi$, iff $\mathfrak{M} \models \varphi$ for every structure $\mathfrak{M}$.

Definition syn.2 (Entailment)**.** A set of sentences Γ *entails* a sentence φ, $\Gamma \vDash \varphi$, iff for every structure $\mathfrak{M}$ with $\mathfrak{M} \models \Gamma$, $\mathfrak{M} \models \varphi$.

Definition syn.3 (Satisfiability)**.** A set of sentences Γ is *satisfiable* if $\mathfrak{M} \models \Gamma$ for some structure $\mathfrak{M}$. If Γ is not satisfiable it is called *unsatisfiable*.

Proposition syn.4. *A sentence φ is valid iff $\Gamma \vDash \varphi$ for every set of sentences Γ.*

Proof. For the forward direction, let φ be valid, and let Γ be a set of sentences. Let $\mathfrak{M}$ be a structure so that $\mathfrak{M} \models \Gamma$. Since φ is valid, $\mathfrak{M} \models \varphi$, hence $\Gamma \vDash \varphi$.

For the contrapositive of the reverse direction, let φ be invalid, so there is a structure $\mathfrak{M}$ with $\mathfrak{M} \not\models \varphi$. When $\Gamma = \{\top\}$, since $\top$ is valid, $\mathfrak{M} \models \Gamma$. Hence, there is a structure $\mathfrak{M}$ so that $\mathfrak{M} \models \Gamma$ but $\mathfrak{M} \not\models \varphi$, hence Γ does not entail φ. □

fol:syn:sem:prop:entails-unsat

Proposition syn.5. *$\Gamma \vDash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable.*

Proof. For the forward direction, suppose $\Gamma \vDash \varphi$ and suppose to the contrary that there is a structure $\mathfrak{M}$ so that $\mathfrak{M} \models \Gamma \cup \{\neg\varphi\}$. Since $\mathfrak{M} \models \Gamma$ and $\Gamma \vDash \varphi$, $\mathfrak{M} \models \varphi$. Also, since $\mathfrak{M} \models \Gamma \cup \{\neg\varphi\}$, $\mathfrak{M} \models \neg\varphi$, so we have both $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \not\models \varphi$, a contradiction. Hence, there can be no such structure $\mathfrak{M}$, so $\Gamma \cup \{\varphi\}$ is unsatisfiable.

For the reverse direction, suppose $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable. So for every structure $\mathfrak{M}$, either $\mathfrak{M} \not\models \Gamma$ or $\mathfrak{M} \models \varphi$. Hence, for every structure $\mathfrak{M}$ with $\mathfrak{M} \models \Gamma$, $\mathfrak{M} \models \varphi$, so $\Gamma \vDash \varphi$. □

Problem syn.1.

1. Show that $\Gamma \vDash \bot$ iff Γ is unsatisfiable.
2. Show that $\Gamma \cup \{\varphi\} \vDash \bot$ iff $\Gamma \vDash \neg\varphi$.
3. Suppose c does not occur in φ or Γ. Show that $\Gamma \vDash \forall x\, \varphi$ iff $\Gamma \vDash \varphi[c/x]$.

Proposition syn.6. *If $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash \varphi$, then $\Gamma' \vDash \varphi$.*

Proof. Suppose that $\Gamma \subseteq \Gamma'$ and $\Gamma \vDash \varphi$. Let $\mathfrak{M}$ be such that $\mathfrak{M} \models \Gamma'$; then $\mathfrak{M} \models \Gamma$, and since $\Gamma \vDash \varphi$, we get that $\mathfrak{M} \models \varphi$. Hence, whenever $\mathfrak{M} \models \Gamma'$, $\mathfrak{M} \models \varphi$, so $\Gamma' \vDash \varphi$. □

Theorem syn.7 (Semantic Deduction Theorem). $\Gamma \cup \{\varphi\} \vDash \psi$ *iff* $\Gamma \vDash \varphi \to \psi$. *fol:syn:sem: thm:sem-deduction*

Proof. For the forward direction, let $\Gamma \cup \{\varphi\} \vDash \psi$ and let $\mathfrak{M}$ be a structure so that $\mathfrak{M} \models \Gamma$. If $\mathfrak{M} \models \varphi$, then $\mathfrak{M} \models \Gamma \cup \{\varphi\}$, so since $\Gamma \cup \{\varphi\}$ entails ψ, we get $\mathfrak{M} \models \psi$. Therefore, $\mathfrak{M} \models \varphi \to \psi$, so $\Gamma \vDash \varphi \to \psi$.

For the reverse direction, let $\Gamma \vDash \varphi \to \psi$ and $\mathfrak{M}$ be a structure so that $\mathfrak{M} \models \Gamma \cup \{\varphi\}$. Then $\mathfrak{M} \models \Gamma$, so $\mathfrak{M} \models \varphi \to \psi$, and since $\mathfrak{M} \models \varphi$, $\mathfrak{M} \models \psi$. Hence, whenever $\mathfrak{M} \models \Gamma \cup \{\varphi\}$, $\mathfrak{M} \models \psi$, so $\Gamma \cup \{\varphi\} \vDash \psi$. □

Proposition syn.8. *Let $\mathfrak{M}$ be a structure, and $\varphi(x)$ a formula with one free variable x, and t a closed term. Then:* *fol:syn:sem: prop:quant-terms*

1. $\varphi(t) \vDash \exists x\, \varphi(x)$
2. $\forall x\, \varphi(x) \vDash \varphi(t)$

Proof.

1. Suppose $\mathfrak{M} \models \varphi(t)$. Let s be a variable assignment with $s(x) = \mathrm{Val}^{\mathfrak{M}}(t)$. Then $\mathfrak{M}, s \models \varphi(t)$ since $\varphi(t)$ is a sentence. By **??**, $\mathfrak{M}, s \models \varphi(x)$. By **??**, $\mathfrak{M} \models \exists x\, \varphi(x)$.
2. Suppose $\mathfrak{M} \models \forall x\, \varphi(x)$. Let s be a variable assignment with $s(x) = \mathrm{Val}^{\mathfrak{M}}(t)$. By **??**, $\mathfrak{M}, s \models \varphi(x)$. By **??**, $\mathfrak{M}, s \models \varphi(t)$. By **??**, $\mathfrak{M} \models \varphi(t)$ since $\varphi(t)$ is a sentence.

□

Problem syn.2. Complete the proof of Proposition syn.8.

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Bibliography