syn.1 Main operator of a Formula

It is often useful to talk about the last operator used in constructing a formula φ. This operator is called the *main operator* of φ. Intuitively, it is the "outermost" operator of φ. For example, the main operator of $\neg\varphi$ is $\neg$, the main operator of $(\varphi \vee \psi)$ is $\vee$, etc.

Definition syn.1 (Main operator). The *main operator* of a formula φ is defined as follows:

1. φ is atomic: φ has no main operator.

2. $\varphi \equiv \neg\psi$: the main operator of φ is $\neg$.

3. $\varphi \equiv (\psi \wedge \chi)$: the main operator of φ is $\wedge$.

4. $\varphi \equiv (\psi \vee \chi)$: the main operator of φ is $\vee$.

5. $\varphi \equiv (\psi \to \chi)$: the main operator of φ is $\to$.

6. $\varphi \equiv (\psi \leftrightarrow \chi)$: the main operator of φ is $\leftrightarrow$.

7. $\varphi \equiv \forall x\,\psi$: the main operator of φ is $\forall$.

8. $\varphi \equiv \exists x\,\psi$: the main operator of φ is $\exists$.

In each case, we intend the specific indicated *occurrence* of the main operator in the formula. For instance, since the formula $((\theta \to \alpha) \to (\alpha \to \theta))$ is of the form $(\psi \to \chi)$ where ψ is $(\theta \to \alpha)$ and χ is $(\alpha \to \theta)$, the second occurrence of $\to$ is the main operator.

This is a *recursive* definition of a function which maps all non-atomic formulas to their main operator occurrence. Because of the way formulas are defined inductively, every formula φ satisfies one of the cases in Definition syn.1. This guarantees that for each non-atomic formula φ a main operator exists. Because each formula satisfies only one of these conditions, and because the smaller formulas from which φ is constructed are uniquely determined in each case, the main operator occurrence of φ is unique, and so we have defined a function.

We call formulas by the names in Table 1 depending on which symbol their main operator is.

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Bibliography

Main operator	Type of formula	Example
none	atomic (formula)	$\bot$, $\top$, $R(t_1, \ldots, t_n)$, $t_1 = t_2$
$\neg$	negation	$\neg\varphi$
$\wedge$	conjunction	$(\varphi \wedge \psi)$
$\vee$	disjunction	$(\varphi \vee \psi)$
$\rightarrow$	conditional	$(\varphi \rightarrow \psi)$
$\leftrightarrow$	biconditional	$(\varphi \leftrightarrow \psi)$
$\forall$	universal (formula)	$\forall x\, \varphi$
$\exists$	existential (formula)	$\exists x\, \varphi$

Table 1: Main operator and names of formulas