syn.1 Main operator of a Formula

fol:syn:mai:sec It is often useful to talk about the last operator used in constructing a formula φ. This operator is called the *main operator* of φ. Intuitively, it is the "outermost" operator of φ. For example, the main operator of $\neg\varphi$ is $\neg$, the main operator of $(\varphi \lor \psi)$ is $\lor$, etc. explanation

fol:syn:mai:def:main-op **Definition syn.1** (Main operator)**.** The *main operator* of a formula φ is defined as follows:

1. φ is atomic: φ has no main operator.
2. $\varphi \equiv \neg\psi$: the main operator of φ is $\neg$.
3. $\varphi \equiv (\psi \land \chi)$: the main operator of φ is $\land$.
4. $\varphi \equiv (\psi \lor \chi)$: the main operator of φ is $\lor$.
5. $\varphi \equiv (\psi \to \chi)$: the main operator of φ is $\to$.
6. $\varphi \equiv (\psi \leftrightarrow \chi)$: the main operator of φ is $\leftrightarrow$.
7. $\varphi \equiv \forall x\, \psi$: the main operator of φ is $\forall$.
8. $\varphi \equiv \exists x\, \psi$: the main operator of φ is $\exists$.

In each case, we intend the specific indicated *occurrence* of the main operator in the formula. For instance, since the formula $((\theta \to \alpha) \to (\alpha \to \theta))$ is of the form $(\psi \to \chi)$ where ψ is $(\theta \to \alpha)$ and χ is $(\alpha \to \theta)$, the second occurrence of $\to$ is the main operator.

This is a *recursive* definition of a function which maps all non-atomic formulas to their main operator occurrence. Because of the way formulas are defined inductively, every formula φ satisfies one of the cases in Definition syn.1. This guarantees that for each non-atomic formula φ a main operator exists. Because each formula satisfies only one of these conditions, and because the smaller formulas from which φ is constructed are uniquely determined in each case, the main operator occurrence of φ is unique, and so we have defined a function. explanation

We call formulas by the following names depending on which symbol their main operator is:

Main operator	Type of formula	Example
none	atomic (formula)	$\bot$, $\top$, $R(t_1, \dots, t_n)$, $t_1 = t_2$
$\neg$	negation	$\neg\varphi$
$\land$	conjunction	$(\varphi \land \psi)$
$\lor$	disjunction	$(\varphi \lor \psi)$
$\to$	conditional	$(\varphi \to \psi)$
$\forall$	universal (formula)	$\forall x\, \varphi$
$\exists$	existential (formula)	$\exists x\, \varphi$

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Bibliography