syn.1 Free Variables and Sentences

fol:syn:fvs:sec

fol:syn:fvs:defn:free-occ **Definition syn.1** (Free occurrences of a variable)**.** The *free* occurrences of a variable in a formula are defined inductively as follows:

1. φ is atomic: all variable occurrences in φ are free.
2. $\varphi \equiv \neg\psi$: the free variable occurrences of φ are exactly those of ψ.
3. $\varphi \equiv (\psi * \chi)$: the free variable occurrences of φ are those in ψ together with those in χ.
4. $\varphi \equiv \forall x\, \psi$: the free variable occurrences in φ are all of those in ψ except for occurrences of x.
5. $\varphi \equiv \exists x\, \psi$: the free variable occurrences in φ are all of those in ψ except for occurrences of x.

Definition syn.2 (Bound Variables)**.** An occurrence of a variable in a formula φ is *bound* if it is not free.

Problem syn.1. Give an inductive definition of the bound variable occurrences along the lines of Definition syn.1.

Definition syn.3 (Scope)**.** If $\forall x\, \psi$ is an occurrence of a subformula in a formula φ, then the corresponding occurrence of ψ in φ is called the *scope* of the corresponding occurrence of $\forall x$. Similarly for $\exists x$.

If ψ is the scope of a quantifier occurrence $\forall x$ or $\exists x$ in φ, then all occurrences of x which are free in ψ are said to be *bound by* the mentioned quantifier occurrence.

Example syn.4. Consider the following formula:

$$\exists \mathsf{v}_0 \; \underbrace{\mathsf{A}^2_0(\mathsf{v}_0, \mathsf{v}_1)}_{\psi}$$

ψ represents the scope of $\exists \mathsf{v}_0$. The quantifier binds the occurence of v_0 in ψ, but does not bind the occurence of v_1. So v_1 is a free variable in this case.

We can now see how this might work in a more complicated formula φ:

$$\forall \mathsf{v}_0 \; \underbrace{(\mathsf{A}^1_0(\mathsf{v}_0) \to \mathsf{A}^2_0(\mathsf{v}_0, \mathsf{v}_1))}_{\psi} \to \exists \mathsf{v}_1 \; \underbrace{(\mathsf{A}^2_1(\mathsf{v}_0, \mathsf{v}_1) \lor \forall \mathsf{v}_0 \; \overbrace{\neg \mathsf{A}^1_1(\mathsf{v}_0)}^{\theta})}_{\chi}$$

ψ is the scope of the first $\forall \mathsf{v}_0$, χ is the scope of $\exists \mathsf{v}_1$, and θ is the scope of the second $\forall \mathsf{v}_0$. The first $\forall \mathsf{v}_0$ binds the occurrences of v_0 in ψ, $\exists \mathsf{v}_1$ the occurrence of v_1 in χ, and the second $\forall \mathsf{v}_0$ binds the occurrence of v_0 in θ. The first occurrence of v_1 and the fourth occurrence of v_0 are free in φ. The last occurrence of v_0 is free in θ, but bound in χ and φ.

Definition syn.5 (Sentence)**.** A formula φ is a *sentence* iff it contains no free occurrences of variables.

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Bibliography