

syn.1 Extensionality

fol:syn:ext:sec Extensionality, sometimes called relevance, can be expressed informally as follows: the only factors that bear upon the satisfaction of **formula** φ in a **structure** \mathfrak{M} relative to a **variable** assignment s , are the size of the **domain** and the assignments made by \mathfrak{M} and s to the elements of the language that actually appear in φ . explanation

One immediate consequence of extensionality is that where two **structures** \mathfrak{M} and \mathfrak{M}' agree on all the elements of the language appearing in a sentence φ and have the same domain, \mathfrak{M} and \mathfrak{M}' must also agree on whether or not φ itself is true.

fol:syn:ext:prop:extensionality **Proposition syn.1 (Extensionality).** *Let φ be a formula, and \mathfrak{M}_1 and \mathfrak{M}_2 be structures with $|\mathfrak{M}_1| = |\mathfrak{M}_2|$, and s a variable assignment on $|\mathfrak{M}_1| = |\mathfrak{M}_2|$. If $c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2}$, $R^{\mathfrak{M}_1} = R^{\mathfrak{M}_2}$, and $f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2}$ for every constant symbol c , relation symbol R , and function symbol f occurring in φ , then $\mathfrak{M}_1, s \models \varphi$ iff $\mathfrak{M}_2, s \models \varphi$.*

Proof. First prove (by induction on t) that for every term, $\text{Val}_s^{\mathfrak{M}_1}(t) = \text{Val}_s^{\mathfrak{M}_2}(t)$. Then prove the proposition by induction on φ , making use of the claim just proved for the induction basis (where φ is atomic). \square

Problem syn.1. Carry out the proof of **Proposition syn.1** in detail.

fol:syn:ext:cor:extensionality-sent **Corollary syn.2 (Extensionality for Sentences).** *Let φ be a sentence and $\mathfrak{M}_1, \mathfrak{M}_2$ as in **Proposition syn.1**. Then $\mathfrak{M}_1 \models \varphi$ iff $\mathfrak{M}_2 \models \varphi$.*

Proof. Follows from **Proposition syn.1** by ?? \square

Moreover, the value of a term, and whether or not a **structure** satisfies a **formula**, only depend on the values of its subterms.

fol:syn:ext:prop:ext-terms **Proposition syn.3.** *Let \mathfrak{M} be a structure, t and t' terms, and s a variable assignment. Then $\text{Val}_s^{\mathfrak{M}}(t[t'/x]) = \text{Val}_{s[\text{Val}_s^{\mathfrak{M}}(t')/x]}^{\mathfrak{M}}(t)$.*

Proof. By induction on t .

1. If t is a constant, say, $t \equiv c$, then $t[t'/x] = c$, and $\text{Val}_s^{\mathfrak{M}}(c) = c^{\mathfrak{M}} = \text{Val}_{s[\text{Val}_s^{\mathfrak{M}}(t')/x]}^{\mathfrak{M}}(c)$.
2. If t is a variable other than x , say, $t \equiv y$, then $t[t'/x] = y$, and $\text{Val}_s^{\mathfrak{M}}(y) = \text{Val}_{s[\text{Val}_s^{\mathfrak{M}}(t')/x]}^{\mathfrak{M}}(y)$ since $s \sim_x s[\text{Val}_s^{\mathfrak{M}}(t')/x]$.
3. If $t \equiv x$, then $t[t'/x] = t'$. But $\text{Val}_{s[\text{Val}_s^{\mathfrak{M}}(t')/x]}^{\mathfrak{M}}(x) = \text{Val}_s^{\mathfrak{M}}(t')$ by definition of $s[\text{Val}_s^{\mathfrak{M}}(t')/x]$.

4. If $t \equiv f(t_1, \dots, t_n)$ then we have:

$$\begin{aligned}
\text{Val}_s^{\mathfrak{M}}(t[t'/x]) &= \\
&= \text{Val}_s^{\mathfrak{M}}(f(t_1[t'/x], \dots, t_n[t'/x])) \\
&\quad \text{by definition of } t[t'/x] \\
&= f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(t_1[t'/x]), \dots, \text{Val}_s^{\mathfrak{M}}(t_n[t'/x])) \\
&\quad \text{by definition of } \text{Val}_s^{\mathfrak{M}}(f(\dots)) \\
&= f^{\mathfrak{M}}(\text{Val}_{s[\text{Val}_s^{\mathfrak{M}}(t')/x]}^{\mathfrak{M}}(t_1), \dots, \text{Val}_{s[\text{Val}_s^{\mathfrak{M}}(t')/x]}^{\mathfrak{M}}(t_n)) \\
&\quad \text{by induction hypothesis} \\
&= \text{Val}_{s[\text{Val}_s^{\mathfrak{M}}(t')/x]}^{\mathfrak{M}}(t) \text{ by definition of } \text{Val}_{s[\text{Val}_s^{\mathfrak{M}}(t')/x]}^{\mathfrak{M}}(f(\dots)) \quad \square
\end{aligned}$$

Proposition syn.4. *Let \mathfrak{M} be a structure, φ a formula, t' a term, and s a variable assignment. Then $\mathfrak{M}, s \models \varphi[t'/x]$ iff $\mathfrak{M}, s[\text{Val}_s^{\mathfrak{M}}(t')/x] \models \varphi$.* fol:syn:ext:
prop:ext-formulas

Proof. Exercise. □

Problem syn.2. Prove **Proposition syn.4**

explanation

The point of **Propositions syn.3** and **syn.4** is the following. Suppose we have a term t or a formula φ and some term t' , and we want to know the value of $t[t'/x]$ or whether or not $\varphi[t'/x]$ is satisfied in a structure \mathfrak{M} relative to a variable assignment s . Then we can either perform the substitution first and then consider the value or satisfaction relative to \mathfrak{M} and s , or we can first determine the value $m = \text{Val}_s^{\mathfrak{M}}(t')$ of t' in \mathfrak{M} relative to s , change the variable assignment to $s[m/x]$ and then consider the value of t in \mathfrak{M} and $s[m/x]$, or whether $\mathfrak{M}, s[m/x] \models \varphi$. **Propositions syn.3** and **syn.4** guarantee that the answer will be the same, whichever way we do it.

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Bibliography