

## syn.1 Extensionality

fol:syn:ext:  
sec Extensionality, sometimes called relevance, can be expressed informally as explanation follows: the only thing that bears upon the satisfaction of **formula**  $\varphi$  in a **structure**  $\mathfrak{M}$  relative to a **variable** assignment  $s$ , are the assignments made by  $\mathfrak{M}$  and  $s$  to the elements of the language that actually appear in  $\varphi$ .

One immediate consequence of extensionality is that where two **structures**  $\mathfrak{M}$  and  $\mathfrak{M}'$  agree on all the elements of the language appearing in a sentence  $\varphi$  and have the same domain,  $\mathfrak{M}$  and  $\mathfrak{M}'$  must also agree on whether or not  $\varphi$  itself is true.

fol:syn:ext:  
prop:extensionality **Proposition syn.1** (Extensionality). *Let  $\varphi$  be a formula, and  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  be structures with  $|\mathfrak{M}_1| = |\mathfrak{M}_2|$ , and  $s$  a variable assignment on  $|\mathfrak{M}_1| = |\mathfrak{M}_2|$ . If  $c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2}$ ,  $R^{\mathfrak{M}_1} = R^{\mathfrak{M}_2}$ , and  $f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2}$  for every constant symbol  $c$ , relation symbol  $R$ , and function symbol  $f$  occurring in  $\varphi$ , then  $\mathfrak{M}_1, s \models \varphi$  iff  $\mathfrak{M}_2, s \models \varphi$ .*

*Proof.* First prove (by induction on  $t$ ) that for every term,  $\text{Val}_s^{\mathfrak{M}_1}(t) = \text{Val}_s^{\mathfrak{M}_2}(t)$ . Then prove the proposition by induction on  $\varphi$ , making use of the claim just proved for the induction basis (where  $\varphi$  is atomic).  $\square$

**Problem syn.1.** Carry out the proof of [Proposition syn.1](#) in detail.

fol:syn:ext:  
cor:extensionality-sent **Corollary syn.2** (Extensionality for Sentences). *Let  $\varphi$  be a sentence and  $\mathfrak{M}_1, \mathfrak{M}_2$  as in [Proposition syn.1](#). Then  $\mathfrak{M}_1 \models \varphi$  iff  $\mathfrak{M}_2 \models \varphi$ .*

*Proof.* Follows from [Proposition syn.1](#) by ???.  $\square$

Moreover, the value of a term, and whether or not a **structure** satisfies a **formula**, only depends on the values of its subterms.

fol:syn:ext:  
prop:ext-terms **Proposition syn.3.** *Let  $\mathfrak{M}$  be a structure,  $t$  and  $t'$  terms, and  $s$  a variable assignment. Let  $s' \sim_x s$  be the  $x$ -variant of  $s$  given by  $s'(x) = \text{Val}_s^{\mathfrak{M}}(t')$ . Then  $\text{Val}_s^{\mathfrak{M}}(t[t'/x]) = \text{Val}_{s'}^{\mathfrak{M}}(t)$ .*

*Proof.* By induction on  $t$ .

1. If  $t$  is a constant, say,  $t \equiv c$ , then  $t[t'/x] = c$ , and  $\text{Val}_s^{\mathfrak{M}}(c) = c^{\mathfrak{M}} = \text{Val}_{s'}^{\mathfrak{M}}(c)$ .
2. If  $t$  is a variable other than  $x$ , say,  $t \equiv y$ , then  $t[t'/x] = y$ , and  $\text{Val}_s^{\mathfrak{M}}(y) = \text{Val}_{s'}^{\mathfrak{M}}(y)$  since  $s' \sim_x s$ .
3. If  $t \equiv x$ , then  $t[t'/x] = t'$ . But  $\text{Val}_s^{\mathfrak{M}}(x) = \text{Val}_s^{\mathfrak{M}}(t')$  by definition of  $s'$ .

4. If  $t \equiv f(t_1, \dots, t_n)$  then we have:

$$\begin{aligned}
 \text{Val}_s^{\mathfrak{M}}(t[t'/x]) &= \\
 &= \text{Val}_s^{\mathfrak{M}}(f(t_1[t'/x], \dots, t_n[t'/x])) \\
 &\quad \text{by definition of } t[t'/x] \\
 &= f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(t_1[t'/x]), \dots, \text{Val}_s^{\mathfrak{M}}(t_n[t'/x])) \\
 &\quad \text{by definition of } \text{Val}_s^{\mathfrak{M}}(f(\dots)) \\
 &= f^{\mathfrak{M}}(\text{Val}_{s'}^{\mathfrak{M}}(t_1), \dots, \text{Val}_{s'}^{\mathfrak{M}}(t_n)) \\
 &\quad \text{by induction hypothesis} \\
 &= \text{Val}_{s'}^{\mathfrak{M}}(t) \text{ by definition of } \text{Val}_{s'}^{\mathfrak{M}}(f(\dots))
 \end{aligned}$$

□

**Proposition syn.4.** Let  $\mathfrak{M}$  be a *structure*,  $\varphi$  a *formula*,  $t$  a *term*, and  $s$  a fol:syn:ext: *variable assignment*. Let  $s' \sim_x s$  be the  $x$ -variant of  $s$  given by  $s'(x) = \text{Val}_s^{\mathfrak{M}}(t)$ . prop:ext-formulas  
Then  $\mathfrak{M}, s \models \varphi[t/x]$  iff  $\mathfrak{M}, s' \models \varphi$ .

*Proof.* Exercise.

□

**Problem syn.2.** Prove [Proposition syn.4](#)

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## Bibliography