

syn.1 Extensionality

fol:syn:ext:sec Extensionality, sometimes called relevance, can be expressed informally as explanation follows: the only factors that bears upon the satisfaction of **formula** φ in a **structure** \mathfrak{M} relative to a **variable** assignment s , are the size of the **domain** and the assignments made by \mathfrak{M} and s to the elements of the language that actually appear in φ .

One immediate consequence of extensionality is that where two **structures** \mathfrak{M} and \mathfrak{M}' agree on all the elements of the language appearing in a sentence φ and have the same domain, \mathfrak{M} and \mathfrak{M}' must also agree on whether or not φ itself is true.

fol:syn:ext:prop:extensionality **Proposition syn.1** (Extensionality). *Let φ be a formula, and \mathfrak{M}_1 and \mathfrak{M}_2 be structures with $|\mathfrak{M}_1| = |\mathfrak{M}_2|$, and s a variable assignment on $|\mathfrak{M}_1| = |\mathfrak{M}_2|$. If $c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2}$, $R^{\mathfrak{M}_1} = R^{\mathfrak{M}_2}$, and $f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2}$ for every constant symbol c , relation symbol R , and function symbol f occurring in φ , then $\mathfrak{M}_1, s \models \varphi$ iff $\mathfrak{M}_2, s \models \varphi$.*

Proof. First prove (by induction on t) that for every term, $\text{Val}_s^{\mathfrak{M}_1}(t) = \text{Val}_s^{\mathfrak{M}_2}(t)$. Then prove the proposition by induction on φ , making use of the claim just proved for the induction basis (where φ is atomic). \square

Problem syn.1. Carry out the proof of [Proposition syn.1](#) in detail.

fol:syn:ext:cor:extensionality-sent **Corollary syn.2** (Extensionality for Sentences). *Let φ be a sentence and $\mathfrak{M}_1, \mathfrak{M}_2$ as in [Proposition syn.1](#). Then $\mathfrak{M}_1 \models \varphi$ iff $\mathfrak{M}_2 \models \varphi$.*

Proof. Follows from [Proposition syn.1](#) by ???. \square

Moreover, the value of a term, and whether or not a **structure** satisfies a **formula**, only depends on the values of its subterms.

fol:syn:ext:prop:ext-terms **Proposition syn.3.** *Let \mathfrak{M} be a structure, t and t' terms, and s a variable assignment. Let $s' \sim_x s$ be the x -variant of s given by $s'(x) = \text{Val}_s^{\mathfrak{M}}(t')$. Then $\text{Val}_s^{\mathfrak{M}}(t[t'/x]) = \text{Val}_{s'}^{\mathfrak{M}}(t)$.*

Proof. By induction on t .

1. If t is a constant, say, $t \equiv c$, then $t[t'/x] = c$, and $\text{Val}_s^{\mathfrak{M}}(c) = c^{\mathfrak{M}} = \text{Val}_{s'}^{\mathfrak{M}}(c)$.
2. If t is a variable other than x , say, $t \equiv y$, then $t[t'/x] = y$, and $\text{Val}_s^{\mathfrak{M}}(y) = \text{Val}_{s'}^{\mathfrak{M}}(y)$ since $s' \sim_x s$.
3. If $t \equiv x$, then $t[t'/x] = t'$. But $\text{Val}_s^{\mathfrak{M}}(x) = \text{Val}_s^{\mathfrak{M}}(t')$ by definition of s' .

4. If $t \equiv f(t_1, \dots, t_n)$ then we have:

$$\begin{aligned}
 \text{Val}_s^{\mathfrak{M}}(t[t'/x]) &= \\
 &= \text{Val}_s^{\mathfrak{M}}(f(t_1[t'/x], \dots, t_n[t'/x])) \\
 &\quad \text{by definition of } t[t'/x] \\
 &= f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(t_1[t'/x]), \dots, \text{Val}_s^{\mathfrak{M}}(t_n[t'/x])) \\
 &\quad \text{by definition of } \text{Val}_s^{\mathfrak{M}}(f(\dots)) \\
 &= f^{\mathfrak{M}}(\text{Val}_{s'}^{\mathfrak{M}}(t_1), \dots, \text{Val}_{s'}^{\mathfrak{M}}(t_n)) \\
 &\quad \text{by induction hypothesis} \\
 &= \text{Val}_{s'}^{\mathfrak{M}}(t) \text{ by definition of } \text{Val}_{s'}^{\mathfrak{M}}(f(\dots))
 \end{aligned}$$

□

Proposition syn.4. Let \mathfrak{M} be a *structure*, φ a *formula*, t a *term*, and s a fol:syn:ext: *variable assignment*. Let $s' \sim_x s$ be the x -variant of s given by $s'(x) = \text{Val}_s^{\mathfrak{M}}(t)$. prop:ext-formulas
Then $\mathfrak{M}, s \models \varphi[t/x]$ iff $\mathfrak{M}, s' \models \varphi$.

Proof. Exercise.

□

Problem syn.2. Prove [Proposition syn.4](#)

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Bibliography