Recall that a term is closed if it contains no variables.

Definition syn.1 (Value of closed terms). If $t$ is a closed term of the language $L$ and $\mathcal{M}$ is a structure for $L$, the value $\text{Val}^\mathcal{M}(t)$ is defined as follows:

1. If $t$ is just the constant symbol $c$, then $\text{Val}^\mathcal{M}(c) = c^\mathcal{M}$.
2. If $t$ is of the form $f(t_1, \ldots, t_n)$, then $\text{Val}^\mathcal{M}(t) = f^\mathcal{M}(\text{Val}^\mathcal{M}(t_1), \ldots, \text{Val}^\mathcal{M}(t_n))$.

Definition syn.2 (Covered structure). A structure is covered if every element of the domain is the value of some closed term.

Example syn.3. Let $L$ be the language with constant symbols zero, one, two, . . . , the binary predicate symbol $<$, and the binary function symbols + and $\times$. Then a structure $\mathcal{M}$ for $L$ is the one with domain $|\mathcal{M}| = \{0, 1, 2, \ldots\}$ and assignments $\text{zero}^\mathcal{M} = 0$, $\text{one}^\mathcal{M} = 1$, $\text{two}^\mathcal{M} = 2$, and so forth. For the binary relation symbol $<$, the set $<^\mathcal{M}$ is the set of all pairs $\langle c_1, c_2 \rangle \in |\mathcal{M}|^2$ such that $c_1$ is less than $c_2$: for example, $\langle 1, 3 \rangle \in <^\mathcal{M}$ but $\langle 2, 2 \rangle \notin <^\mathcal{M}$. For the binary function symbol $+$, define $+^\mathcal{M}$ in the usual way—for example, $+^\mathcal{M}(2, 3)$ maps to 5, and similarly for the binary function symbol $\times$. Hence, the value of four is just 4, and the value of $\times(two, +(three, zero))$ (or in infix notation, $two \times (three + zero)$) is

$$\text{Val}^\mathcal{M}(\times(two, +(three, zero))) = \times^\mathcal{M}(\text{Val}^\mathcal{M}(two), \text{Val}^\mathcal{M}(two, +(three, zero)))$$
$$= \times^\mathcal{M}(\text{Val}^\mathcal{M}(two), +(\text{Val}^\mathcal{M}(three), \text{Val}^\mathcal{M}(zero)))$$
$$= \times^\mathcal{M}(two^\mathcal{M}, +(three^\mathcal{M}, zero^\mathcal{M}))$$
$$= \times^\mathcal{M}(2, +(3, 0))$$
$$= \times^\mathcal{M}(2, 3)$$
$$= 6$$

Problem syn.1. Is $\mathcal{M}$, the standard model of arithmetic, covered? Explain.

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Bibliography