syn.1 Covered Structures for First-order Languages

Recall that a term is \textit{closed} if it contains no variables.

**Definition syn.1 (Value of closed terms).** If \( t \) is a closed term of the language \( \mathcal{L} \) and \( \mathfrak{M} \) is a structure for \( \mathcal{L} \), the \textit{value} \( \text{Val}^{\mathfrak{M}}(t) \) is defined as follows:

1. If \( t \) is just the constant symbol \( c \), then \( \text{Val}^{\mathfrak{M}}(c) = c^{\mathfrak{M}} \).
2. If \( t \) is of the form \( f(t_1, \ldots, t_n) \), then
   \[
   \text{Val}^{\mathfrak{M}}(t) = f^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(t_1), \ldots, \text{Val}^{\mathfrak{M}}(t_n)).
   \]

**Definition syn.2 (Covered structure).** A structure is \textit{covered} if every element of the domain is the value of some closed term.

**Example syn.3.** Let \( \mathcal{L} \) be the language with constant symbols \textit{zero}, \textit{one}, \textit{two}, \ldots, the binary predicate symbol \textit{<}, and the binary function symbols \textit{+} and \textit{×}. Then a structure \( \mathfrak{M} \) for \( \mathcal{L} \) is the one with domain \( |\mathfrak{M}| = \{0, 1, 2, \ldots\} \) and assignments \( \text{zero}^{\mathfrak{M}} = 0, \text{one}^{\mathfrak{M}} = 1, \text{two}^{\mathfrak{M}} = 2 \), and so forth. For the binary relation symbol \textit{<}, the set \( <^{\mathfrak{M}} \) is the set of all pairs \( \langle c_1, c_2 \rangle \in |\mathfrak{M}|^2 \) such that \( c_1 \) is less than \( c_2 \); for example, \( \langle 1, 3 \rangle \in <^{\mathfrak{M}} \) but \( \langle 2, 2 \rangle \notin <^{\mathfrak{M}} \). For the binary function symbol \textit{+}, define \( +^{\mathfrak{M}} \) in the usual way—for example, \( +^{\mathfrak{M}}(2, 3) \) maps to 5, and similarly for the binary function symbol \textit{×}. Hence, the value of \textit{four} is just 4, and the value of \( \times(two, +(three, zero)) \) (or in infix notation, \( two \times (three + zero) \)) is

\[
\text{Val}^{\mathfrak{M}}(\times(two, +(three, zero))) = \\
\quad = \times^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(two), \text{Val}^{\mathfrak{M}}(two, +(three, zero))) \\
\quad = \times^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(two), +^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(three), \text{Val}^{\mathfrak{M}}(zero))) \\
\quad = \times^{\mathfrak{M}}(two^{\mathfrak{M}}, +^{\mathfrak{M}}(three^{\mathfrak{M}}, zero^{\mathfrak{M}})) \\
\quad = \times^{\mathfrak{M}}(2, +^{\mathfrak{M}}(3, 0)) \\
\quad = \times^{\mathfrak{M}}(2, 3) \\
\quad = 6
\]

**Problem syn.1.** Is \( \mathfrak{M} \), the standard model of arithmetic, covered? Explain.

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Bibliography