

syn.1 Variable Assignments

fol:syn:ass: sec A **variable** assignment s provides a value for *every* variable—and there are explanation infinitely many of them. This is of course not necessary. We require **variable** assignments to assign values to all **variables** simply because it makes things a lot easier. The value of a term t , and whether or not a **formula** φ is satisfied in a **structure** with respect to s , only depend on the assignments s makes to the **variables** in t and the free **variables** of φ . This is the content of the next two propositions. To make the idea of “depends on” precise, we show that any two variable assignments that agree on all the variables in t give the same value, and that φ is satisfied relative to one iff it is satisfied relative to the other if two variable assignments agree on all free variables of φ .

fol:syn:ass: prop:valindep **Proposition syn.1.** *If the **variables** in a term t are among x_1, \dots, x_n , and $s_1(x_i) = s_2(x_i)$ for $i = 1, \dots, n$, then $\text{Val}_{s_1}^{\mathfrak{M}}(t) = \text{Val}_{s_2}^{\mathfrak{M}}(t)$.*

Proof. By induction on the complexity of t . For the base case, t can be a **constant symbol** or one of the variables x_1, \dots, x_n . If $t = c$, then $\text{Val}_{s_1}^{\mathfrak{M}}(t) = c^{\mathfrak{M}} = \text{Val}_{s_2}^{\mathfrak{M}}(t)$. If $t = x_i$, $s_1(x_i) = s_2(x_i)$ by the hypothesis of the proposition, and so $\text{Val}_{s_1}^{\mathfrak{M}}(t) = s_1(x_i) = s_2(x_i) = \text{Val}_{s_2}^{\mathfrak{M}}(t)$.

For the inductive step, assume that $t = f(t_1, \dots, t_k)$ and that the claim holds for t_1, \dots, t_k . Then

$$\begin{aligned} \text{Val}_{s_1}^{\mathfrak{M}}(t) &= \text{Val}_{s_1}^{\mathfrak{M}}(f(t_1, \dots, t_k)) = \\ &= f^{\mathfrak{M}}(\text{Val}_{s_1}^{\mathfrak{M}}(t_1), \dots, \text{Val}_{s_1}^{\mathfrak{M}}(t_k)) \end{aligned}$$

For $j = 1, \dots, k$, the **variables** of t_j are among x_1, \dots, x_n . So by induction hypothesis, $\text{Val}_{s_1}^{\mathfrak{M}}(t_j) = \text{Val}_{s_2}^{\mathfrak{M}}(t_j)$. So,

$$\begin{aligned} \text{Val}_{s_1}^{\mathfrak{M}}(t) &= \text{Val}_{s_2}^{\mathfrak{M}}(f(t_1, \dots, t_k)) = \\ &= f^{\mathfrak{M}}(\text{Val}_{s_1}^{\mathfrak{M}}(t_1), \dots, \text{Val}_{s_1}^{\mathfrak{M}}(t_k)) = \\ &= f^{\mathfrak{M}}(\text{Val}_{s_2}^{\mathfrak{M}}(t_1), \dots, \text{Val}_{s_2}^{\mathfrak{M}}(t_k)) = \\ &= \text{Val}_{s_2}^{\mathfrak{M}}(f(t_1, \dots, t_k)) = \text{Val}_{s_2}^{\mathfrak{M}}(t). \end{aligned}$$

□

fol:syn:ass: prop:satindep **Proposition syn.2.** *If the free **variables** in φ are among x_1, \dots, x_n , and $s_1(x_i) = s_2(x_i)$ for $i = 1, \dots, n$, then $\mathfrak{M}, s_1 \models \varphi$ iff $\mathfrak{M}, s_2 \models \varphi$.*

Proof. We use induction on the complexity of φ . For the base case, where φ is atomic, φ can be: \top , \perp , $R(t_1, \dots, t_k)$ for a k -place predicate R and terms t_1, \dots, t_k , or $t_1 = t_2$ for terms t_1 and t_2 .

1. $\varphi \equiv \top$: both $\mathfrak{M}, s_1 \models \varphi$ and $\mathfrak{M}, s_2 \models \varphi$.
2. $\varphi \equiv \perp$: both $\mathfrak{M}, s_1 \not\models \varphi$ and $\mathfrak{M}, s_2 \not\models \varphi$.

3. $\varphi \equiv R(t_1, \dots, t_k)$: let $\mathfrak{M}, s_1 \models \varphi$. Then

$$\langle \text{Val}_{s_1}^{\mathfrak{M}}(t_1), \dots, \text{Val}_{s_1}^{\mathfrak{M}}(t_k) \rangle \in R^{\mathfrak{M}}.$$

For $i = 1, \dots, k$, $\text{Val}_{s_1}^{\mathfrak{M}}(t_i) = \text{Val}_{s_2}^{\mathfrak{M}}(t_i)$ by [Proposition syn.1](#). So we also have $\langle \text{Val}_{s_2}^{\mathfrak{M}}(t_1), \dots, \text{Val}_{s_2}^{\mathfrak{M}}(t_k) \rangle \in R^{\mathfrak{M}}$.

4. $\varphi \equiv t_1 = t_2$: suppose $\mathfrak{M}, s_1 \models \varphi$. Then $\text{Val}_{s_1}^{\mathfrak{M}}(t_1) = \text{Val}_{s_1}^{\mathfrak{M}}(t_2)$. So,

$$\begin{aligned} \text{Val}_{s_2}^{\mathfrak{M}}(t_1) &= \text{Val}_{s_1}^{\mathfrak{M}}(t_1) && \text{(by Proposition syn.1)} \\ &= \text{Val}_{s_1}^{\mathfrak{M}}(t_2) && \text{(since } \mathfrak{M}, s_1 \models t_1 = t_2 \text{)} \\ &= \text{Val}_{s_2}^{\mathfrak{M}}(t_2) && \text{(by Proposition syn.1),} \end{aligned}$$

so $\mathfrak{M}, s_2 \models t_1 = t_2$.

Now assume $\mathfrak{M}, s_1 \models \psi$ iff $\mathfrak{M}, s_2 \models \psi$ for all [formulas](#) ψ less complex than φ . The induction step proceeds by cases determined by the main operator of φ . In each case, we only demonstrate the forward direction of the [biconditional](#); the proof of the reverse direction is symmetrical. In all cases except those for the quantifiers, we apply the induction hypothesis to sub-[formulas](#) ψ of φ . The free variables of ψ are among those of φ . Thus, if s_1 and s_2 agree on the free variables of φ , they also agree on those of ψ , and the induction hypothesis applies to ψ .

1. $\varphi \equiv \neg\psi$: if $\mathfrak{M}, s_1 \models \varphi$, then $\mathfrak{M}, s_1 \not\models \psi$, so by the induction hypothesis, $\mathfrak{M}, s_2 \not\models \psi$, hence $\mathfrak{M}, s_2 \models \varphi$.
2. $\varphi \equiv \psi \wedge \chi$: if $\mathfrak{M}, s_1 \models \varphi$, then $\mathfrak{M}, s_1 \models \psi$ and $\mathfrak{M}, s_1 \models \chi$, so by induction hypothesis, $\mathfrak{M}, s_2 \models \psi$ and $\mathfrak{M}, s_2 \models \chi$. Hence, $\mathfrak{M}, s_2 \models \varphi$.
3. $\varphi \equiv \psi \vee \chi$: if $\mathfrak{M}, s_1 \models \varphi$, then $\mathfrak{M}, s_1 \models \psi$ or $\mathfrak{M}, s_1 \models \chi$. By induction hypothesis, $\mathfrak{M}, s_2 \models \psi$ or $\mathfrak{M}, s_2 \models \chi$, so $\mathfrak{M}, s_2 \models \varphi$.
4. $\varphi \equiv \psi \rightarrow \chi$: if $\mathfrak{M}, s_1 \models \varphi$, then $\mathfrak{M}, s_1 \not\models \psi$ or $\mathfrak{M}, s_1 \models \chi$. By the induction hypothesis, $\mathfrak{M}, s_2 \not\models \psi$ or $\mathfrak{M}, s_2 \models \chi$, so $\mathfrak{M}, s_2 \models \varphi$.
5. $\varphi \equiv \psi \leftrightarrow \chi$: if $\mathfrak{M}, s_1 \models \varphi$, then either $\mathfrak{M}, s_1 \models \psi$ and $\mathfrak{M}, s_1 \models \chi$, or $\mathfrak{M}, s_1 \not\models \psi$ and $\mathfrak{M}, s_1 \not\models \chi$. By the induction hypothesis, either $\mathfrak{M}, s_2 \models \psi$ and $\mathfrak{M}, s_2 \models \chi$ or $\mathfrak{M}, s_2 \not\models \psi$ and $\mathfrak{M}, s_2 \not\models \chi$. In either case, $\mathfrak{M}, s_2 \models \varphi$.
6. $\varphi \equiv \exists x \psi$: if $\mathfrak{M}, s_1 \models \varphi$, there is an x -variant s'_1 of s_1 so that $\mathfrak{M}, s'_1 \models \psi$. Let s'_2 be the x -variant of s_2 that assigns the same thing to x as does s'_1 . The free variables of ψ are among x_1, \dots, x_n , and x . $s'_1(x_i) = s'_2(x_i)$, since s'_1 and s'_2 are x -variants of s_1 and s_2 , respectively, and by hypothesis $s_1(x_i) = s_2(x_i)$. $s'_1(x) = s'_2(x)$ by the way we have defined s'_2 . Then the induction hypothesis applies to ψ and s'_1, s'_2 , so $\mathfrak{M}, s'_2 \models \psi$. Hence, there is an x -variant of s_2 that satisfies ψ , and so $\mathfrak{M}, s_2 \models \varphi$.

7. $\varphi \equiv \forall x \psi$: if $\mathfrak{M}, s_1 \models \varphi$, then for every x -variant s'_1 of s_1 , $\mathfrak{M}, s'_1 \models \psi$.
 Take an arbitrary x -variant s'_2 of s_2 , let s'_1 be the x -variant of s_1 which assigns the same thing to x as does s'_2 . The free variables of ψ are among x_1, \dots, x_n , and x . $s'_1(x_i) = s'_2(x_i)$, since s'_1 and s'_2 are x -variants of s_1 and s_2 , respectively, and by hypothesis $s_1(x_i) = s_2(x_i)$. $s'_1(x) = s'_2(x)$ by the way we have defined s'_1 . Then the induction hypothesis applies to ψ and s'_1, s_2 , and we have $\mathfrak{M}, s'_2 \models \psi$. Since s'_2 is an arbitrary x -variant of s_2 , every x -variant of s_2 satisfies ψ , and so $\mathfrak{M}, s_2 \models \varphi$.

By induction, we get that $\mathfrak{M}, s_1 \models \varphi$ iff $\mathfrak{M}, s_2 \models \varphi$ whenever the free variables in φ are among x_1, \dots, x_n and $s_1(x_i) = s_2(x_i)$ for $i = 1, \dots, n$. \square

Problem syn.1. Complete the proof of Proposition syn.2.

Sentences have no free variables, so any two variable assignments assign the same things to all the (zero) free variables of any sentence. The proposition explanation just proved then means that whether or not a sentence is satisfied in a structure relative to a variable assignment is completely independent of the assignment. We'll record this fact. It justifies the definition of satisfaction of a sentence in a structure (without mentioning a variable assignment) that follows.

fol:syn:ass: cor:sat-sentence **Corollary syn.3.** If φ is a sentence and s a variable assignment, then $\mathfrak{M}, s \models \varphi$ iff $\mathfrak{M}, s' \models \varphi$ for every variable assignment s' .

Proof. Let s' be any variable assignment. Since φ is a sentence, it has no free variables, and so every variable assignment s' trivially assigns the same things to all free variables of φ as does s . So the condition of Proposition syn.2 is satisfied, and we have $\mathfrak{M}, s \models \varphi$ iff $\mathfrak{M}, s' \models \varphi$. \square

fol:syn:ass: defn:satisfaction **Definition syn.4.** If φ is a sentence, we say that a structure \mathfrak{M} satisfies φ , $\mathfrak{M} \models \varphi$, iff $\mathfrak{M}, s \models \varphi$ for all variable assignments s .

If $\mathfrak{M} \models \varphi$, we also simply say that φ is true in \mathfrak{M} .

fol:syn:ass: prop:sentence-sat-true **Proposition syn.5.** Let \mathfrak{M} be a structure, φ be a sentence, and s a variable assignment. $\mathfrak{M} \models \varphi$ iff $\mathfrak{M}, s \models \varphi$.

Proof. Exercise. \square

Problem syn.2. Prove Proposition syn.5

fol:syn:ass: prop:sat-quant **Proposition syn.6.** Suppose $\varphi(x)$ only contains x free, and \mathfrak{M} is a structure. Then:

1. $\mathfrak{M} \models \exists x \varphi(x)$ iff $\mathfrak{M}, s \models \varphi(x)$ for at least one variable assignment s .
2. $\mathfrak{M} \models \forall x \varphi(x)$ iff $\mathfrak{M}, s \models \varphi(x)$ for all variable assignments s .

Proof. Exercise. \square

Problem syn.3. Prove Proposition syn.6.

Problem syn.4. Suppose \mathcal{L} is a language without function symbols. Given a structure \mathfrak{M} , c a constant symbol and $a \in |\mathfrak{M}|$, define $\mathfrak{M}[a/c]$ to be the structure that is just like \mathfrak{M} , except that $c^{\mathfrak{M}[a/c]} = a$. Define $\mathfrak{M} \models \varphi$ for sentences φ by:

1. $\varphi \equiv \perp$: not $\mathfrak{M} \models \varphi$.
2. $\varphi \equiv \top$: $\mathfrak{M} \models \varphi$.
3. $\varphi \equiv R(d_1, \dots, d_n)$: $\mathfrak{M} \models \varphi$ iff $\langle d_1^{\mathfrak{M}}, \dots, d_n^{\mathfrak{M}} \rangle \in R^{\mathfrak{M}}$.
4. $\varphi \equiv d_1 = d_2$: $\mathfrak{M} \models \varphi$ iff $d_1^{\mathfrak{M}} = d_2^{\mathfrak{M}}$.
5. $\varphi \equiv \neg\psi$: $\mathfrak{M} \models \varphi$ iff not $\mathfrak{M} \models \psi$.
6. $\varphi \equiv (\psi \wedge \chi)$: $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \psi$ and $\mathfrak{M} \models \chi$.
7. $\varphi \equiv (\psi \vee \chi)$: $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \psi$ or $\mathfrak{M} \models \chi$ (or both).
8. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathfrak{M} \models \varphi$ iff not $\mathfrak{M} \models \psi$ or $\mathfrak{M} \models \chi$ (or both).
9. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{M} \models \varphi$ iff either both $\mathfrak{M} \models \psi$ and $\mathfrak{M} \models \chi$, or neither $\mathfrak{M} \models \psi$ nor $\mathfrak{M} \models \chi$.
10. $\varphi \equiv \forall x \psi$: $\mathfrak{M} \models \varphi$ iff for all $a \in |\mathfrak{M}|$, $\mathfrak{M}[a/c] \models \psi[c/x]$, if c does not occur in ψ .
11. $\varphi \equiv \exists x \psi$: $\mathfrak{M} \models \varphi$ iff there is an $a \in |\mathfrak{M}|$ such that $\mathfrak{M}[a/c] \models \psi[c/x]$, if c does not occur in ψ .

Let x_1, \dots, x_n be all free variables in φ , c_1, \dots, c_n constant symbols not in φ , $a_1, \dots, a_n \in |\mathfrak{M}|$, and $s(x_i) = a_i$.

Show that $\mathfrak{M}, s \models \varphi$ iff $\mathfrak{M}[a_1/c_1, \dots, a_n/c_n] \models \varphi[c_1/x_1] \dots [c_n/x_n]$.

(This problem shows that it is possible to give a semantics for first-order logic that makes do without variable assignments.)

Problem syn.5. Suppose that f is a function symbol not in $\varphi(x, y)$. Show that there is a structure \mathfrak{M} such that $\mathfrak{M} \models \forall x \exists y \varphi(x, y)$ iff there is an \mathfrak{M}' such that $\mathfrak{M}' \models \forall x \varphi(x, f(x))$.

(This problem is a special case of what's known as Skolem's Theorem; $\forall x \varphi(x, f(x))$ is called a *Skolem normal form* of $\forall x \exists y \varphi(x, y)$.)

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Bibliography