

## seq.1 Soundness

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sec

A **derivation** system, such as the sequent calculus, is *sound* if it cannot **derive** things that do not actually hold. Soundness is thus a kind of guaranteed safety property for **derivation** systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

explanation

1. every **derivable sentence** is valid;
2. if a **sentence** is **derivable** from some others, it is also a consequence of them;
3. if a set of **sentences** is inconsistent, it is unsatisfiable.

These are important properties of a **derivation** system. If any of them do not hold, the **derivation** system is deficient—it would **derive** too much. Consequently, establishing the soundness of a **derivation** system is of the utmost importance.

Because all these proof-theoretic properties are defined via **derivability** in the sequent calculus of certain sequents, proving (1)–(3) above requires proving something about the semantic properties of **derivable** sequents. We will first define what it means for a sequent to be *valid*, and then show that every **derivable** sequent is valid. (1)–(3) then follow as corollaries from this result.

**Definition seq.1.** A **structure**  $\mathfrak{M}$  *satisfies* a sequent  $\Gamma \Rightarrow \Delta$  iff either  $\mathfrak{M} \not\models \varphi$  for some  $\varphi \in \Gamma$  or  $\mathfrak{M} \models \varphi$  for some  $\varphi \in \Delta$ .

A sequent is *valid* iff every **structure**  $\mathfrak{M}$  satisfies it.

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thm:sequent-soundness

**Theorem seq.2** (Soundness). *If LK derives  $\Theta \Rightarrow \Xi$ , then  $\Theta \Rightarrow \Xi$  is valid.*

*Proof.* Let  $\pi$  be a **derivation** of  $\Theta \Rightarrow \Xi$ . We proceed by induction on the number of inferences  $n$  in  $\pi$ .

If the number of inferences is 0, then  $\pi$  consists only of an initial sequent. Every initial sequent  $\varphi \Rightarrow \varphi$  is obviously valid, since for every  $\mathfrak{M}$ , either  $\mathfrak{M} \not\models \varphi$  or  $\mathfrak{M} \models \varphi$ .

If the number of inferences is greater than 0, we distinguish cases according to the type of the lowermost inference. By induction hypothesis, we can assume that the premises of that inference are valid, since the number of inferences in the proof of any premise is smaller than  $n$ .

First, we consider the possible inferences with only one premise.

1. The last inference is a weakening. Then  $\Theta \Rightarrow \Xi$  is either  $A, \Gamma \Rightarrow \Delta$  (if the last inference is WL) or  $\Gamma \Rightarrow \Delta, \varphi$  (if it's WR), and the **derivation** ends in one of

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta \end{array}}{\varphi, \Gamma \Rightarrow \Delta} \text{WL} \qquad \frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta \end{array}}{\Gamma \Rightarrow \Delta, \varphi} \text{WR}$$

By induction hypothesis,  $\Gamma \Rightarrow \Delta$  is valid, i.e., for every **structure**  $\mathfrak{M}$ , either there is some  $\chi \in \Gamma$  such that  $\mathfrak{M} \not\models \chi$  or there is some  $\chi \in \Delta$  such that  $\mathfrak{M} \models \chi$ .

If  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , then  $\chi \in \Theta$  as well since  $\Theta = \varphi, \Gamma$ , and so  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Theta$ . Similarly, if  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ , as  $\chi \in \Xi$ ,  $\mathfrak{M} \models \chi$  for some  $\chi \in \Xi$ . Consequently,  $\Theta \Rightarrow \Xi$  is valid.

2. The last inference is  $\neg$ L: Then the premise of the last inference is  $\Gamma \Rightarrow \Delta, \varphi$  and the conclusion is  $\neg\varphi, \Gamma \Rightarrow \Delta$ , i.e., the **derivation** ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array}}{\neg\varphi, \Gamma \Rightarrow \Delta} \neg\text{L}$$

and  $\Theta = \neg\varphi, \Gamma$  while  $\Xi = \Delta$ .

The induction hypothesis tells us that  $\Gamma \Rightarrow \Delta, \varphi$  is valid, i.e., for every  $\mathfrak{M}$ , either (a) for some  $\chi \in \Gamma$ ,  $\mathfrak{M} \not\models \chi$ , or (b) for some  $\chi \in \Delta$ ,  $\mathfrak{M} \models \chi$ , or (c)  $\mathfrak{M} \models \varphi$ . We want to show that  $\Theta \Rightarrow \Xi$  is also valid. Let  $\mathfrak{M}$  be a **structure**. If (a) holds, then there is  $\chi \in \Gamma$  so that  $\mathfrak{M} \not\models \varphi$ , but  $\varphi \in \Theta$  as well. If (b) holds, there is  $\chi \in \Delta$  such that  $\mathfrak{M} \models \chi$ , but  $\chi \in \Xi$  as well. Finally, if  $\mathfrak{M} \models \varphi$ , then  $\mathfrak{M} \not\models \neg\varphi$ . Since  $\neg\varphi \in \Theta$ , there is  $\chi \in \Theta$  such that  $\mathfrak{M} \not\models \chi$ . Consequently,  $\Theta \Rightarrow \Xi$  is valid.

3. The last inference is  $\neg$ R: Exercise.
4. The last inference is  $\wedge$ L: There are two variants:  $\varphi \wedge \psi$  may be inferred on the left from  $\varphi$  or from  $\psi$  on the left side of the premise. In the first case, the  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \varphi, \Gamma \Rightarrow \Delta \end{array}}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\text{L}$$

and  $\Theta = \varphi \wedge \psi, \Gamma$  while  $\Xi = \Delta$ . Consider a **structure**  $\mathfrak{M}$ . Since by induction hypothesis,  $\varphi, \Gamma \Rightarrow \Delta$  is valid, (a)  $\mathfrak{M} \not\models \varphi$ , (b)  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , or (c)  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ . In case (a),  $\mathfrak{M} \not\models \varphi \wedge \psi$ , so there is  $\chi \in \Theta$  (namely,  $\varphi \wedge \psi$ ) such that  $\mathfrak{M} \not\models \chi$ . In case (b), there is  $\chi \in \Theta$  such that  $\mathfrak{M} \not\models \chi$ , and  $\chi \in \Theta$  as well. In case (c), there is  $\chi \in \Delta$  such that  $\mathfrak{M} \models \chi$ , and  $\chi \in \Xi$  as well since  $\Xi = \Delta$ . So in each case,  $\mathfrak{M}$  satisfies  $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$ . Since  $\mathfrak{M}$  was arbitrary,  $\Gamma \Rightarrow \Delta$  is valid. The case where  $\varphi \wedge \psi$  is inferred from  $\psi$  is handled the same, changing  $\varphi$  to  $\psi$ .

5. The last inference is  $\vee\text{R}$ : There are two variants:  $\varphi \vee \psi$  may be inferred on the right from  $\varphi$  or from  $\psi$  on the right side of the premise. In the first case,  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array}}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \vee\text{R}$$

Now  $\Theta = \Gamma$  and  $\Xi = \Delta, \varphi \vee \psi$ . Consider a **structure**  $\mathfrak{M}$ . Since  $\Gamma \Rightarrow \Delta, \varphi$  is valid, (a)  $\mathfrak{M} \models \varphi$ , (b)  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , or (c)  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ . In case (a),  $\mathfrak{M} \models \varphi \vee \psi$ . In case (b), there is  $\chi \in \Gamma$  such that  $\mathfrak{M} \not\models \chi$ . In case (c), there is  $\chi \in \Delta$  such that  $\mathfrak{M} \models \chi$ . So in each case,  $\mathfrak{M}$  satisfies  $\Gamma \Rightarrow \Delta, \varphi \vee \psi$ , i.e.,  $\Theta \Rightarrow \Xi$ . Since  $\mathfrak{M}$  was arbitrary,  $\Theta \Rightarrow \Xi$  is valid. The case where  $\varphi \vee \psi$  is inferred from  $\psi$  is handled the same, changing  $\varphi$  to  $\psi$ .

6. The last inference is  $\rightarrow\text{R}$ : Then  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \varphi, \Gamma \Rightarrow \Delta, \varphi \end{array}}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \rightarrow\text{R}$$

Again, the induction hypothesis says that the premise is valid; we want to show that the conclusion is valid as well. Let  $\mathfrak{M}$  be arbitrary. Since  $\varphi, \Gamma \Rightarrow \Delta, \psi$  is valid, at least one of the following cases obtains: (a)  $\mathfrak{M} \not\models \varphi$ , (b)  $\mathfrak{M} \models \psi$ , (c)  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , or (d)  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ . In cases (a) and (b),  $\mathfrak{M} \models \varphi \rightarrow \psi$  and so there is a  $\chi \in \Delta, \varphi \rightarrow \psi$  such that  $\mathfrak{M} \models \chi$ . In case (c), for some  $\chi \in \Gamma$ ,  $\mathfrak{M} \not\models \chi$ . In case (d), for some  $\chi \in \Delta$ ,  $\mathfrak{M} \models \chi$ . In each case,  $\mathfrak{M}$  satisfies  $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$ . Since  $\mathfrak{M}$  was arbitrary,  $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$  is valid.

7. The last inference is  $\forall\text{L}$ : Then there is a **formula**  $\varphi(x)$  and a closed term  $t$  such that  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \varphi(t), \Gamma \Rightarrow \Delta \end{array}}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \forall\text{L}$$

We want to show that the conclusion  $\forall x \varphi(x), \Gamma \Rightarrow \Delta$  is valid. Consider a **structure**  $\mathfrak{M}$ . Since the premise  $\varphi(t), \Gamma \Rightarrow \Delta$  is valid, (a)  $\mathfrak{M} \not\models \varphi(t)$ , (b)  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , or (c)  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ . In case (a),

by ??, if  $\mathfrak{M} \models \forall x \varphi(x)$ , then  $\mathfrak{M} \models \varphi(t)$ . Since  $\mathfrak{M} \not\models \varphi(t)$ ,  $\mathfrak{M} \not\models \forall x \varphi(x)$ . In case (b) and (c),  $\mathfrak{M}$  also satisfies  $\forall x \varphi(x), \Gamma \Rightarrow \Delta$ . Since  $\mathfrak{M}$  was arbitrary,  $\forall x \varphi(x), \Gamma \Rightarrow \Delta$  is valid.

8. The last inference is  $\exists R$ : Exercise.
9. The last inference is  $\forall R$ : Then there is a formula  $\varphi(x)$  and a constant symbol  $a$  such that  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi(a) \end{array}}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} \forall R$$

where the eigenvariable condition is satisfied, i.e.,  $a$  does not occur in  $\varphi(x)$ ,  $\Gamma$ , or  $\Delta$ . By induction hypothesis, the premise of the last inference is valid. We have to show that the conclusion is valid as well, i.e., that for any structure  $\mathfrak{M}$ , (a)  $\mathfrak{M} \models \forall x \varphi(x)$ , (b)  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , or (c)  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ .

Suppose  $\mathfrak{M}$  is an arbitrary structure. If (b) or (c) holds, we are done, so suppose neither holds: for all  $\chi \in \Gamma$ ,  $\mathfrak{M} \models \chi$ , and for all  $\chi \in \Delta$ ,  $\mathfrak{M} \not\models \chi$ . We have to show that (a) holds, i.e.,  $\mathfrak{M} \models \forall x \varphi(x)$ . By ??, it suffices to show that  $\mathfrak{M}, s \models \varphi(x)$  for all variable assignments  $s$ . So let  $s$  be an arbitrary variable assignment. Consider the structure  $\mathfrak{M}'$  which is just like  $\mathfrak{M}$  except  $a^{\mathfrak{M}'} = s(x)$ . By ??, for any  $\chi \in \Gamma$ ,  $\mathfrak{M}' \models \chi$  since  $a$  does not occur in  $\Gamma$ , and for any  $\chi \in \Delta$ ,  $\mathfrak{M}' \not\models \chi$ . But the premise is valid, so  $\mathfrak{M}' \models \varphi(a)$ . By ??,  $\mathfrak{M}', s \models \varphi(a)$ , since  $\varphi(a)$  is a sentence. Now  $s \sim_x s$  with  $s(x) = \text{Val}_s^{\mathfrak{M}'}(a)$ , since we've defined  $\mathfrak{M}'$  in just this way. So ?? applies, and we get  $\mathfrak{M}', s \models \varphi(x)$ . Since  $a$  does not occur in  $\varphi(x)$ , by ??,  $\mathfrak{M}, s \models \varphi(x)$ . Since  $s$  was arbitrary, we've completed the proof that  $\mathfrak{M}, s \models \varphi(x)$  for all variable assignments.

10. The last inference is  $\exists L$ : Exercise.

Now let's consider the possible inferences with two premises.

1. The last inference is a cut: then  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \varphi, \Pi \Rightarrow \Lambda \end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \text{Cut}$$

Let  $\mathfrak{M}$  be a structure. By induction hypothesis, the premises are valid, so  $\mathfrak{M}$  satisfies both premises. We distinguish two cases: (a)  $\mathfrak{M} \not\models \varphi$  and

(b)  $\mathfrak{M} \models \varphi$ . In case (a), in order for  $\mathfrak{M}$  to satisfy the left premise, it must satisfy  $\Gamma \Rightarrow \Delta$ . But then it also satisfies the conclusion. In case (b), in order for  $\mathfrak{M}$  to satisfy the right premise, it must satisfy  $\Pi \setminus \Delta$ . Again,  $\mathfrak{M}$  satisfies the conclusion.

2. The last inference is  $\wedge R$ . Then  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \psi \end{array}}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge R$$

Consider a **structure**  $\mathfrak{M}$ . If  $\mathfrak{M}$  satisfies  $\Gamma \Rightarrow \Delta$ , we are done. So suppose it doesn't. Since  $\Gamma \Rightarrow \Delta, \varphi$  is valid by induction hypothesis,  $\mathfrak{M} \models \varphi$ . Similarly, since  $\Gamma \Rightarrow \Delta, \psi$  is valid,  $\mathfrak{M} \models \psi$ . But then  $\mathfrak{M} \models \varphi \wedge \psi$ .

3. The last inference is  $\vee L$ : Exercise.

4. The last inference is  $\rightarrow L$ . Then  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ \psi, \Pi \Rightarrow \Delta \end{array}}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Delta} \rightarrow L$$

Again, consider a **structure**  $\mathfrak{M}$  and suppose  $\mathfrak{M}$  doesn't satisfy  $\Gamma, \Pi \Rightarrow \Delta, \Delta$ . We have to show that  $\mathfrak{M} \not\models \varphi \rightarrow \psi$ . If  $\mathfrak{M}$  doesn't satisfy  $\Gamma, \Pi \Rightarrow \Delta, \Delta$ , it satisfies neither  $\Gamma \Rightarrow \Delta$  nor  $\Pi \Rightarrow \Delta$ . Since  $\Gamma \Rightarrow \Delta, \varphi$  is valid, we have  $\mathfrak{M} \models \varphi$ . Since  $\psi, \Pi \Rightarrow \Delta$  is valid, we have  $\mathfrak{M} \not\models \psi$ . But then  $\mathfrak{M} \not\models \varphi \rightarrow \psi$ , which is what we wanted to show.

□

**Problem seq.1.** Complete the proof of [Theorem seq.2](#).

*fol:seq:sou:* **Corollary seq.3.** *cor:weak-soundness* If  $\vdash \varphi$  then  $\varphi$  is valid.

*fol:seq:sou:* **Corollary seq.4.** *cor:entailment-soundness* If  $\Gamma \vdash \varphi$  then  $\Gamma \models \varphi$ .

*Proof.* If  $\Gamma \vdash \varphi$  then for some finite subset  $\Gamma_0 \subseteq \Gamma$ , there is a **derivation** of  $\Gamma_0 \Rightarrow \varphi$ . By [Theorem seq.2](#), every **structure**  $\mathfrak{M}$  either makes some  $\psi \in \Gamma_0$  false or makes  $\varphi$  true. Hence, if  $\mathfrak{M} \models \Gamma$  then also  $\mathfrak{M} \models \varphi$ . □

*fol:seq:sou:* **Corollary seq.5.** *cor:consistency-soundness* If  $\Gamma$  is satisfiable, then it is consistent.

*Proof.* We prove the contrapositive. Suppose that  $\Gamma$  is not consistent. Then there is a finite  $\Gamma_0 \subseteq \Gamma$  and a derivation of  $\Gamma_0 \Rightarrow \perp$ . By [Theorem seq.2](#),  $\Gamma_0 \Rightarrow \perp$  is valid. In other words, for every structure  $\mathfrak{M}$ , there is  $\chi \in \Gamma_0$  so that  $\mathfrak{M} \not\models \chi$ , and since  $\Gamma_0 \subseteq \Gamma$ , that  $\chi$  is also in  $\Gamma$ . Thus, no  $\mathfrak{M}$  satisfies  $\Gamma$ , and  $\Gamma$  is not satisfiable.  $\square$

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## Bibliography