Soundness with Identity predicate

**Proposition seq.1.** LK with initial sequents and rules for identity is sound.

*Proof.* Initial sequents of the form $\Rightarrow t = t$ are valid, since for every structure $\mathcal{M}$, $\mathcal{M} \vDash t = t$. (Note that we assume the term $t$ to be closed, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a derivation is $\Rightarrow$. Then the premise is $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)$ and the conclusion is $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)$. Consider a structure $\mathcal{M}$. We need to show that the conclusion is valid, i.e., if $\mathcal{M} \vDash t_1 = t_2$ and $\mathcal{M} \vDash \Gamma$, then either $\mathcal{M} \vDash \chi$ for some $\chi \in \Delta$ or $\mathcal{M} \vDash \varphi(t_2)$.

By induction hypothesis, the premise is valid. This means that if $\mathcal{M} \vDash t_1 = t_2$ and $\mathcal{M} \vDash \Gamma$ either (a) for some $\chi \in \Delta$, $\mathcal{M} \vDash \chi$ or (b) $\mathcal{M} \vDash \varphi(t_1)$. In case (a) we are done. Consider case (b). Let $s$ be a variable assignment with $s(x) = \text{Val}_\mathcal{M}(t_1)$. By $??$, $\mathcal{M}, s \vDash \varphi(t_1)$. Since $s \sim_s s$, by $??$, $\mathcal{M}, s \vDash \varphi(x)$. since $\mathcal{M} \vDash t_1 = t_2$, we have $\text{Val}_\mathcal{M}(t_1) = \text{Val}_\mathcal{M}(t_2)$, and hence $s(x) = \text{Val}_\mathcal{M}(t_2)$. By applying $??$ again, we also have $\mathcal{M}, s \vDash \varphi(t_2)$. By $??$, $\mathcal{M} \vDash \varphi(t_2)$.

qed

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Bibliography