seq.1 Soundness with Identity predicate

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Proposition seq.1. **LK** *with initial sequents and rules for identity is sound.*

Proof. Initial sequents of the form $\Rightarrow t = t$ are valid, since for every structure $\mathfrak{M}$, $\mathfrak{M} \vDash t = t$. (Note that we assume the term t to be closed, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a derivation is $=$. Then the premise is $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)$ and the conclusion is $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)$. Consider a structure $\mathfrak{M}$. We need to show that the conclusion is valid, i.e., if $\mathfrak{M} \vDash t_1 = t_2$ and $\mathfrak{M} \vDash \Gamma$, then either $\mathfrak{M} \vDash \chi$ for some $\chi \in \Delta$ or $\mathfrak{M} \vDash \varphi(t_2)$.

By induction hypothesis, the premise is valid. This means that if $\mathfrak{M} \vDash t_1 = t_2$ and $\mathfrak{M} \vDash \Gamma$ either (a) for some $\chi \in \Delta$, $\mathfrak{M} \vDash \chi$ or (b) $\mathfrak{M} \vDash \varphi(t_1)$. In case (a) we are done. Consider case (b). Let s be a variable assignment with $s(x) = \mathrm{Val}^{\mathfrak{M}}(t_1)$. By **??**, $\mathfrak{M}, s \vDash \varphi(t_1)$. Since $s \sim_x s$, by **??**, $\mathfrak{M}, s \vDash \varphi(x)$. since $\mathfrak{M} \vDash t_1 = t_2$, we have $\mathrm{Val}^{\mathfrak{M}}(t_1) = \mathrm{Val}^{\mathfrak{M}}(t_2)$, and hence $s(x) = \mathrm{Val}^{\mathfrak{M}}(t_2)$. By applying **??** again, we also have $\mathfrak{M}, s \vDash \varphi(t_2)$. By **??**, $\mathfrak{M} \vDash \varphi(t_2)$. $\square$

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Bibliography