seq.1  Soundness with Identity predicate

Proposition seq.1. LK with initial sequents and rules for identity is sound.

Proof. Initial sequents of the form \( \Rightarrow t = t \) are valid, since for every structure \( \mathcal{M} \), \( \mathcal{M} \models t = t \). (Note that we assume the term \( t \) to be closed, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a derivation is \( = \). Then the premise is \( t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1) \) and the conclusion is \( t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2) \). Consider a structure \( \mathcal{M} \). We need to show that the conclusion is valid, i.e., if \( \mathcal{M} \models t_1 = t_2 \) and \( \mathcal{M} \models \Gamma \), then either \( \mathcal{M} \models \chi \) for some \( \chi \in \Delta \) or \( \mathcal{M} \models \varphi(t_2) \).

By induction hypothesis, the premise is valid. This means that if \( \mathcal{M} \models t_1 = t_2 \) and \( \mathcal{M} \models \Gamma \) either (a) for some \( \chi \in \Delta \), \( \mathcal{M} \models \chi \) or (b) \( \mathcal{M} \models \varphi(t_1) \). In case (a) we are done. Consider case (b). Let \( s \) be a variable assignment with \( s(x) = \text{Val}^\mathcal{M}(t_1) \). By ??, \( \mathcal{M}, s \models \varphi(t_1) \). Since \( s \sim_x s \), by ??, \( \mathcal{M}, s \models \varphi(x) \). since \( \mathcal{M} \models t_1 = t_2 \), we have \( \text{Val}^\mathcal{M}(t_1) = \text{Val}^\mathcal{M}(t_2) \), and hence \( s(x) = \text{Val}^\mathcal{M}(t_2) \). By applying ?? again, we also have \( \mathcal{M}, s \models \varphi(t_2) \). By ??, \( \mathcal{M} \models \varphi(t_2) \). \( \square \)

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Bibliography