

seq.1 Soundness with Identity predicate

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Proposition seq.1. **LK** with initial sequents and rules for identity is sound.

Proof. Initial sequents of the form $\Rightarrow t = t$ are valid, since for every **structure** \mathfrak{M} , $\mathfrak{M} \models t = t$. (Note that we assume the term t to be closed, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a **derivation** is $=$. Then the premise is $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)$ and the conclusion is $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)$. Consider a **structure** \mathfrak{M} . We need to show that the conclusion is valid, i.e., if $\mathfrak{M} \models t_1 = t_2$ and $\mathfrak{M} \models \Gamma$, then either $\mathfrak{M} \models \chi$ for some $\chi \in \Delta$ or $\mathfrak{M} \models \varphi(t_2)$.

By induction hypothesis, the premise is valid. This means that if $\mathfrak{M} \models t_1 = t_2$ and $\mathfrak{M} \models \Gamma$ either (a) for some $\chi \in \Delta$, $\mathfrak{M} \models \chi$ or (b) $\mathfrak{M} \models \varphi(t_1)$. In case (a) we are done. Consider case (b). Let s be a variable assignment with $s(x) = \text{Val}^{\mathfrak{M}}(t_1)$. By ??, $\mathfrak{M}, s \models \varphi(t_1)$. Since $s \sim_x s$, by ??, $\mathfrak{M}, s \models \varphi(x)$. since $\mathfrak{M} \models t_1 = t_2$, we have $\text{Val}^{\mathfrak{M}}(t_1) = \text{Val}^{\mathfrak{M}}(t_2)$, and hence $s(x) = \text{Val}^{\mathfrak{M}}(t_2)$. By applying ?? again, we also have $\mathfrak{M}, s \models \varphi(t_2)$. By ??, $\mathfrak{M} \models \varphi(t_2)$. \square

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Bibliography