

# Chapter udf

## The Sequent Calculus

This chapter presents Gentzen’s standard sequent calculus LK for classical first-order logic. It could use more examples and exercises. To include or exclude material relevant to the sequent calculus as a proof system, use the “prfLK” tag.

### seq.1 Rules and Derivations

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sec For the following, let  $\Gamma, \Delta, \Pi, \Lambda$  represent finite sequences of **sentences**.

**Definition seq.1** (Sequent). A *sequent* is an expression of the form

$$\Gamma \Rightarrow \Delta$$

where  $\Gamma$  and  $\Delta$  are finite (possibly empty) sequences of **sentences** of the language  $\mathcal{L}$ .  $\Gamma$  is called the *antecedent*, while  $\Delta$  is the *succedent*.

The intuitive idea behind a sequent is: if all of the **sentences** in the antecedent hold, then at least one of the **sentences** in the succedent holds. That is, if  $\Gamma = \langle \varphi_1, \dots, \varphi_m \rangle$  and  $\Delta = \langle \psi_1, \dots, \psi_n \rangle$ , then  $\Gamma \Rightarrow \Delta$  holds iff explanation

$$(\varphi_1 \wedge \dots \wedge \varphi_m) \rightarrow (\psi_1 \vee \dots \vee \psi_n)$$

holds. There are two special cases: where  $\Gamma$  is empty and when  $\Delta$  is empty. When  $\Gamma$  is empty, i.e.,  $m = 0$ ,  $\Rightarrow \Delta$  holds iff  $\psi_1 \vee \dots \vee \psi_n$  holds. When  $\Delta$  is empty, i.e.,  $n = 0$ ,  $\Gamma \Rightarrow$  holds iff  $\neg(\varphi_1 \wedge \dots \wedge \varphi_m)$  does. We say a sequent is valid iff the corresponding **sentence** is valid.

If  $\Gamma$  is a sequence of **sentences**, we write  $\Gamma, \varphi$  for the result of appending  $\varphi$  to the right end of  $\Gamma$  (and  $\varphi, \Gamma$  for the result of appending  $\varphi$  to the left end of  $\Gamma$ ). If  $\Delta$  is a sequence of **sentences** also, then  $\Gamma, \Delta$  is the concatenation of the two sequences.

**Definition seq.2** (Initial Sequent). An *initial sequent* is a sequent of one of the following forms:

1.  $\varphi \Rightarrow \varphi$
2.  $\Rightarrow \top$
3.  $\perp \Rightarrow$

for any **sentence**  $\varphi$  in the language.

**Derivations** in the sequent calculus are certain trees of sequents, where the topmost sequents are initial sequents, and if a sequent stands below one or two other sequents, it must follow correctly by a rule of inference. The rules for **LK** are divided into two main types: *logical* rules and *structural* rules. The logical rules are named for the **main operator** of the **sentence** containing  $\varphi$  and/or  $\psi$  in the lower sequent. Each one comes in two versions, one for inferring a sequent with the **sentence** containing the **logical operator** on the left, and one with the **sentence** on the right.

## seq.2 Propositional Rules

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### Rules for $\neg$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \neg\text{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} \neg\text{R}$$

### Rules for $\wedge$

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\text{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge\text{R}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\text{L}$$

### Rules for $\vee$

$$\frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \vee\text{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \vee\text{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \vee\text{R}$$

**Rules for  $\rightarrow$**

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Lambda}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \rightarrow\text{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \rightarrow\text{R}$$

### seq.3 Quantifier Rules

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**Rules for  $\forall$**

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \forall\text{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} \forall\text{R}$$

In  $\forall\text{L}$ ,  $t$  is a closed term (i.e., one without variables). In  $\forall\text{R}$ ,  $a$  is a **constant symbol** which must not occur anywhere in the lower sequent of the  $\forall\text{R}$  rule. We call  $a$  the *eigenvariable* of the  $\forall\text{R}$  inference.

**Rules for  $\exists$**

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x \varphi(x), \Gamma \Rightarrow \Delta} \exists\text{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)} \exists\text{R}$$

Again,  $t$  is a closed term, and  $a$  is a **constant symbol** which does not occur in the lower sequent of the  $\exists\text{L}$  rule. We call  $a$  the *eigenvariable* of the  $\exists\text{L}$  inference.

The condition that an eigenvariable not occur in the lower sequent of the  $\forall\text{R}$  or  $\exists\text{L}$  inference is called the *eigenvariable condition*.

We use the term “eigenvariable” even though  $a$  in the above rules is a **constant symbol**. This has historical reasons. - explanation

In  $\exists\text{R}$  and  $\forall\text{L}$  there are no restrictions on the term  $t$ . On the other hand, in the  $\exists\text{L}$  and  $\forall\text{R}$  rules, the eigenvariable condition requires that the **constant**

symbol  $a$  does not occur anywhere outside of  $\varphi(a)$  in the upper sequent. It is necessary to ensure that the system is sound, i.e., only **derives** sequents that are valid. Without this condition, the following would be allowed:

$$\frac{\frac{\varphi(a) \Rightarrow \varphi(a)}{\exists x \varphi(x) \Rightarrow \varphi(a)} * \exists L}{\exists x \varphi(x) \Rightarrow \forall x \varphi(x)} \forall R \qquad \frac{\frac{\varphi(a) \Rightarrow \varphi(a)}{\varphi(a) \Rightarrow \forall x \varphi(x)} * \forall R}{\exists x \varphi(x) \Rightarrow \forall x \varphi(x)} \exists L$$

However,  $\exists x \varphi(x) \Rightarrow \forall x \varphi(x)$  is not valid.

## seq.4 Structural Rules

We also need a few rules that allow us to rearrange **sentences** in the left and right side of a sequent. Since the logical rules require that the **sentences** in the premise which the rule acts upon stand either to the far left or to the far right, we need an “exchange” rule that allows us to move **sentences** to the right position. It’s also important sometimes to be able to combine two identical **sentences** into one, and to add a **sentence** on either side. fol:seq:srl:sec

### Weakening

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{WL} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \text{WR}$$

### Contraction

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{CL} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \text{CR}$$

### Exchange

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \text{XL} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda} \text{XR}$$

A series of weakening, contraction, and exchange inferences will often be indicated by double inference lines.

The following rule, called “cut,” is not strictly speaking necessary, but makes it a lot easier to reuse and combine derivations.

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \text{Cut}$$

## seq.5 Derivations

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We've said what an initial sequent looks like, and we've given the rules of inference. **Derivations** in the sequent calculus are inductively generated from these: each **derivation** either is an initial sequent on its own, or consists of one or two **derivations** followed by an inference. explanation

**Definition seq.3 (LK derivation).** An **LK-derivation** of a sequent  $S$  is a tree of sequents satisfying the following conditions:

1. The topmost sequents of the tree are initial sequents.
2. The bottommost sequent of the tree is  $S$ .
3. Every sequent in the tree except  $S$  is a premise of a correct application of an inference rule whose conclusion stands directly below that sequent in the tree.

We then say that  $S$  is the *end-sequent* of the **derivation** and that  $S$  is *derivable in LK* (or **LK-derivable**).

**Example seq.4.** Every initial sequent, e.g.,  $\chi \Rightarrow \chi$  is a **derivation**. We can obtain a new **derivation** from this by applying, say, the WL rule,

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{WL}$$

The rule, however, is meant to be general: we can replace the  $\varphi$  in the rule with any **sentence**, e.g., also with  $\theta$ . If the premise matches our initial sequent  $\chi \Rightarrow \chi$ , that means that both  $\Gamma$  and  $\Delta$  are just  $\chi$ , and the conclusion would then be  $\theta, \chi \Rightarrow \chi$ . So, the following is a **derivation**:

$$\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}$$

We can now apply another rule, say XL, which allows us to switch two **sentences** on the left. So, the following is also a correct **derivation**:

$$\frac{\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}}{\chi, \theta \Rightarrow \chi} \text{XL}$$

In this application of the rule, which was given as

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \text{XL}$$

both  $\Gamma$  and  $\Pi$  were empty,  $\Delta$  is  $\chi$ , and the roles of  $\varphi$  and  $\psi$  are played by  $\theta$  and  $\chi$ , respectively. In much the same way, we also see that

$$\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{WL}$$

is a **derivation**. Now we can take these two derivations, and combine them using  $\wedge\text{R}$ . That rule was

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge\text{R}$$

In our case, the premises must match the last sequents of the **derivations** ending in the premises. That means that  $\Gamma$  is  $\chi, \theta$ ,  $\Delta$  is empty,  $\varphi$  is  $\chi$  and  $\psi$  is  $\theta$ . So the conclusion, if the inference should be correct, is  $\chi, \theta \Rightarrow \chi \wedge \theta$ . Of course, we can also reverse the premises, then  $\varphi$  would be  $\theta$  and  $\psi$  would be  $\chi$ . So both of the following are correct **derivations**.

$$\frac{\frac{\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}}{\chi, \theta \Rightarrow \chi} \text{XL}}{\chi, \theta \Rightarrow \chi \wedge \theta} \wedge\text{R} \quad \frac{\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{WL}}{\chi, \theta \Rightarrow \theta \wedge \chi} \wedge\text{R}$$

## seq.6 Examples of Derivations

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**Example seq.5.** Give an **LK**-derivation for the sequent  $\varphi \wedge \psi \Rightarrow \varphi$ .

We begin by writing the desired end-sequent at the bottom of the derivation.

$$\overline{\varphi \wedge \psi \Rightarrow \varphi}$$

Next, we need to figure out what kind of inference could have a lower sequent of this form. This could be a structural rule, but it is a good idea to start by looking for a logical rule. The only logical connective occurring in the lower sequent is  $\wedge$ , so we're looking for an  $\wedge$  rule, and since the  $\wedge$  symbol occurs in the antecedent, we're looking at the  $\wedge\text{L}$  rule.

$$\overline{\varphi \wedge \psi \Rightarrow \varphi} \wedge\text{L}$$

There are two options for what could have been the upper sequent of the  $\wedge\text{L}$  inference: we could have an upper sequent of  $\varphi \Rightarrow \varphi$ , or of  $\psi \Rightarrow \varphi$ . Clearly,  $\varphi \Rightarrow \varphi$  is an initial sequent (which is a good thing), while  $\psi \Rightarrow \varphi$  is not derivable in general. We fill in the upper sequent:

$$\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi} \wedge\text{L}$$

We now have a correct **LK**-derivation of the sequent  $\varphi \wedge \psi \Rightarrow \varphi$ .

**Example seq.6.** Give an **LK**-derivation for the sequent  $\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi$ .

Begin by writing the desired end-sequent at the bottom of the derivation.

$$\frac{}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi}$$

To find a logical rule that could give us this end-sequent, we look at the logical connectives in the end-sequent:  $\neg$ ,  $\vee$ , and  $\rightarrow$ . We only care at the moment about  $\vee$  and  $\rightarrow$  because they are **main operators** of **sentences** in the end-sequent, while  $\neg$  is inside the scope of another connective, so we will take care of it later. Our options for logical rules for the final inference are therefore the  $\vee$ L rule and the  $\rightarrow$ R rule. We could pick either rule, really, but let's pick the  $\rightarrow$ R rule (if for no reason other than it allows us to put off splitting into two branches). According to the form of  $\rightarrow$ R inferences which can yield the lower sequent, this must look like:

$$\frac{\frac{}{\varphi, \neg\varphi \vee \psi \Rightarrow \psi}}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi} \rightarrow R$$

If we move  $\neg\varphi \vee \psi$  to the outside of the antecedent, we can apply the  $\vee$ L rule. According to the schema, this must split into two upper sequents as follows:

$$\frac{\frac{\frac{}{\neg\varphi, \varphi \Rightarrow \psi}}{\neg\varphi \vee \psi, \varphi \Rightarrow \psi} \vee L \quad \frac{\frac{}{\psi, \varphi \Rightarrow \psi}}{\varphi, \neg\varphi \vee \psi \Rightarrow \psi} XR}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi} \rightarrow R$$

Remember that we are trying to wind our way up to initial sequents; we seem to be pretty close! The right branch is just one weakening and one exchange away from an initial sequent and then it is done:

$$\frac{\frac{\frac{}{\neg\varphi, \varphi \Rightarrow \psi}}{\neg\varphi \vee \psi, \varphi \Rightarrow \psi} \vee L \quad \frac{\frac{\frac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} WL}{\psi, \varphi \Rightarrow \psi} XL}{\varphi, \neg\varphi \vee \psi \Rightarrow \psi} XR}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi} \rightarrow R$$

Now looking at the left branch, the only logical connective in any **sentence** is the  $\neg$  symbol in the antecedent **sentences**, so we're looking at an instance of the  $\neg$ L rule.

$$\begin{array}{c}
\frac{}{\varphi \Rightarrow \psi, \varphi} \quad \frac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} \text{WL} \\
\frac{}{\neg\varphi, \varphi \Rightarrow \psi} \neg\text{L} \quad \frac{\psi, \varphi \Rightarrow \psi}{\psi, \varphi \Rightarrow \psi} \text{XL} \\
\hline
\frac{}{\neg\varphi \vee \psi, \varphi \Rightarrow \psi} \text{XR} \\
\frac{\varphi, \neg\varphi \vee \psi \Rightarrow \psi}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi} \rightarrow\text{R}
\end{array}$$

Similarly to how we finished off the right branch, we are just one weakening and one exchange away from finishing off this left branch as well.

$$\begin{array}{c}
\frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi, \psi} \text{WR} \\
\frac{\varphi \Rightarrow \psi, \varphi}{\neg\varphi, \varphi \Rightarrow \psi} \neg\text{L} \quad \frac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} \text{WL} \\
\frac{}{\varphi \Rightarrow \psi, \varphi} \text{XR} \quad \frac{\psi, \varphi \Rightarrow \psi}{\psi, \varphi \Rightarrow \psi} \text{XL} \\
\hline
\frac{}{\neg\varphi \vee \psi, \varphi \Rightarrow \psi} \text{XR} \\
\frac{\varphi, \neg\varphi \vee \psi \Rightarrow \psi}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi} \rightarrow\text{R}
\end{array}$$

**Example seq.7.** Give an **LK**-derivation of the sequent  $\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)$

Using the techniques from above, we start by writing the desired end-sequent at the bottom.

$$\overline{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)}$$

The available main connectives of **sentences** in the end-sequent are the  $\vee$  symbol and the  $\neg$  symbol. It would work to apply either the  $\vee\text{L}$  or the  $\neg\text{R}$  rule here, but we start with the  $\neg\text{R}$  rule because it avoids splitting up into two branches for a moment:

$$\frac{\overline{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow}}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \neg\text{R}$$

Now we have a choice of whether to look at the  $\wedge\text{L}$  or the  $\vee\text{L}$  rule. Let's see what happens when we apply the  $\wedge\text{L}$  rule: we have a choice to start with either the sequent  $\varphi, \neg\varphi \vee \neg\psi \Rightarrow$  or the sequent  $\psi, \neg\varphi \vee \neg\psi \Rightarrow$ . Since the proof is symmetric with regards to  $\varphi$  and  $\psi$ , let's go with the former:

$$\frac{\frac{\overline{\varphi, \neg\varphi \vee \neg\psi \Rightarrow}}{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow} \wedge\text{L}}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \neg\text{R}$$

Continuing to fill in the derivation, we see that we run into a problem:

$$\begin{array}{c}
\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg\text{L} \quad \frac{\overline{\varphi \Rightarrow \psi} \quad ?}{\neg\psi, \varphi \Rightarrow} \neg\text{L} \\
\hline
\frac{}{\neg\varphi \vee \neg\psi, \varphi \Rightarrow} \vee\text{L} \\
\frac{\varphi, \neg\varphi \vee \neg\psi \Rightarrow}{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow} \wedge\text{L} \\
\hline
\frac{}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \neg\text{R}
\end{array}$$

The top of the right branch cannot be reduced any further, and it cannot be brought by way of structural inferences to an initial sequent, so this is not the right path to take. So clearly, it was a mistake to apply the  $\wedge$ L rule above. Going back to what we had before and carrying out the  $\vee$ L rule instead, we get

$$\frac{\frac{\frac{\overline{\neg\varphi, \varphi \wedge \psi \Rightarrow}}{\neg\varphi \vee \neg\psi, \varphi \wedge \psi \Rightarrow} \vee\text{L}}{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow} \text{XL}}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \neg\text{R}$$

Completing each branch as we've done before, we get

$$\frac{\frac{\frac{\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi} \wedge\text{L}}{\neg\varphi, \varphi \wedge \psi \Rightarrow} \neg\text{L}}{\neg\varphi \vee \neg\psi, \varphi \wedge \psi \Rightarrow} \vee\text{L}}{\frac{\frac{\frac{\psi \Rightarrow \psi}{\varphi \wedge \psi \Rightarrow \psi} \wedge\text{L}}{\neg\psi, \varphi \wedge \psi \Rightarrow} \neg\text{L}}{\neg\varphi \vee \neg\psi, \varphi \wedge \psi \Rightarrow} \vee\text{L}}{\frac{\frac{\frac{\overline{\neg\varphi \vee \neg\psi, \varphi \wedge \psi \Rightarrow}}{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow} \text{XL}}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \neg\text{R}}$$

(We could have carried out the  $\wedge$  rules lower than the  $\neg$  rules in these steps and still obtained a correct derivation).

**Example seq.8.** So far we haven't used the contraction rule, but it is sometimes required. Here's an example where that happens. Suppose we want to prove  $\Rightarrow A \vee \neg\varphi$ . Applying  $\vee$ R backwards would give us one of these two derivations:

$$\frac{\overline{\Rightarrow \varphi}}{\Rightarrow \varphi \vee \neg\varphi} \vee\text{R} \qquad \frac{\overline{\varphi \Rightarrow}}{\Rightarrow \neg\varphi} \neg\text{R} \qquad \frac{\overline{\varphi \Rightarrow}}{\Rightarrow \varphi \vee \neg\varphi} \vee\text{R}$$

Neither of these of course ends in an initial sequent. The trick is to realize that the contraction rule allows us to combine two copies of a sentence into one—and when we're searching for a proof, i.e., going from bottom to top, we can keep a copy of  $\varphi \vee \neg\varphi$  in the premise, e.g.,

$$\frac{\frac{\overline{\Rightarrow \varphi \vee \neg\varphi, \varphi}}{\Rightarrow \varphi \vee \neg\varphi, \varphi \vee \neg\varphi} \vee\text{R}}{\Rightarrow \varphi \vee \neg\varphi} \text{CR}$$

Now we can apply  $\vee$ R a second time, and also get  $\neg\varphi$ , which leads to a complete derivation.

$$\frac{\frac{\frac{\frac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi} \neg\text{R}}{\Rightarrow \varphi, \varphi \vee \neg\varphi} \vee\text{R}}{\Rightarrow \varphi \vee \neg\varphi, \varphi} \text{XR}}{\frac{\frac{\overline{\Rightarrow \varphi \vee \neg\varphi, \varphi \vee \neg\varphi}}{\Rightarrow \varphi \vee \neg\varphi} \vee\text{R}}{\Rightarrow \varphi \vee \neg\varphi} \text{CR}}$$

**Problem seq.1.** Give **derivations** of the following sequents:

1.  $\Rightarrow \neg(\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \neg\psi)$
2.  $(\varphi \wedge \psi) \rightarrow \chi \Rightarrow (\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)$

## seq.7 Derivations with Quantifiers

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**Example seq.9.** Give an **LK**-derivation of the sequent  $\exists x \neg\varphi(x) \Rightarrow \neg\forall x \varphi(x)$ .

When dealing with quantifiers, we have to make sure not to violate the eigenvariable condition, and sometimes this requires us to play around with the order of carrying out certain inferences. In general, it helps to try and take care of rules subject to the eigenvariable condition first (they will be lower down in the finished proof). Also, it is a good idea to try and look ahead and try to guess what the initial sequent might look like. In our case, it will have to be something like  $\varphi(a) \Rightarrow \varphi(a)$ . That means that when we are “reversing” the quantifier rules, we will have to pick the same term—what we will call  $a$ —for both the  $\forall$  and the  $\exists$  rule. If we picked different terms for each rule, we would end up with something like  $\varphi(a) \Rightarrow \varphi(b)$ , which, of course, is not derivable.

Starting as usual, we write

$$\frac{}{\exists x \neg\varphi(x) \Rightarrow \neg\forall x \varphi(x)}$$

We could either carry out the  $\exists\text{L}$  rule or the  $\neg\text{R}$  rule. Since the  $\exists\text{L}$  rule is subject to the eigenvariable condition, it’s a good idea to take care of it sooner rather than later, so we’ll do that one first.

$$\frac{\frac{}{\neg\varphi(a) \Rightarrow \neg\forall x \varphi(x)}}{\exists x \neg\varphi(x) \Rightarrow \neg\forall x \varphi(x)} \exists\text{L}$$

Applying the  $\neg\text{L}$  and  $\neg\text{R}$  rules backwards, we get

$$\frac{\frac{\frac{\frac{}{\forall x \varphi(x) \Rightarrow \varphi(a)}}{\neg\varphi(a), \forall x \varphi(x) \Rightarrow} \neg\text{L}}{\forall x \varphi(x), \neg\varphi(a) \Rightarrow} \text{XL}}{\frac{\neg\varphi(a) \Rightarrow \neg\forall x \varphi(x)}{\exists x \neg\varphi(x) \Rightarrow \neg\forall x \varphi(x)} \neg\text{R}} \exists\text{L}$$

At this point, our only option is to carry out the  $\forall\text{L}$  rule. Since this rule is not subject to the eigenvariable restriction, we’re in the clear. Remember, we want to try and obtain an initial sequent (of the form  $\varphi(a) \Rightarrow \varphi(a)$ ), so we should choose  $a$  as our argument for  $\varphi$  when we apply the rule.

$$\begin{array}{c}
\frac{\varphi(a) \Rightarrow \varphi(a)}{\forall x \varphi(x) \Rightarrow \varphi(a)} \forall\text{L} \\
\frac{}{\neg\varphi(a), \forall x \varphi(x) \Rightarrow} \neg\text{L} \\
\frac{}{\forall x \varphi(x), \neg\varphi(a) \Rightarrow} \text{XL} \\
\frac{}{\neg\varphi(a) \Rightarrow \neg\forall x \varphi(x)} \neg\text{R} \\
\frac{}{\exists x \neg\varphi(x) \Rightarrow \neg\forall x \varphi(x)} \exists\text{L}
\end{array}$$

It is important, especially when dealing with quantifiers, to double check at this point that the eigenvariable condition has not been violated. Since the only rule we applied that is subject to the eigenvariable condition was  $\exists\text{L}$ , and the eigenvariable  $a$  does not occur in its lower sequent (the end-sequent), this is a correct derivation.

**Problem seq.2.** Give **derivations** of the following sequents:

1.  $\forall x (\varphi(x) \rightarrow \psi) \Rightarrow (\exists y \varphi(y) \rightarrow \psi)$
2.  $\exists x (\varphi(x) \rightarrow \forall y \varphi(y))$

This section collects the definitions of the provability relation and consistency for natural deduction.

## seq.8 Proof-Theoretic Notions

fol:seq:ptn: sec Just as we've defined a number of important semantic notions (validity, explanation entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the **derivability** or **non-derivability** of certain sequents. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorem*.

**Definition seq.10** (Theorems). A **sentence**  $\varphi$  is a *theorem* if there is a **derivation** in **LK** of the sequent  $\Rightarrow \varphi$ . We write  $\vdash \varphi$  if  $\varphi$  is a theorem and  $\not\vdash \varphi$  if it is not.

**Definition seq.11** (Derivability). A **sentence**  $\varphi$  is *derivable* from a set of **sentences**  $\Gamma$ ,  $\Gamma \vdash \varphi$ , iff there is a finite subset  $\Gamma_0 \subseteq \Gamma$  and a sequence  $\Gamma'_0$  of the **sentences** in  $\Gamma_0$  such that **LK** derives  $\Gamma'_0 \Rightarrow \varphi$ . If  $\varphi$  is not **derivable** from  $\Gamma$  we write  $\Gamma \not\vdash \varphi$ .

Because of the contraction, weakening, and exchange rules, the order and number of **sentences** in  $\Gamma'_0$  does not matter: if a sequent  $\Gamma'_0 \Rightarrow \varphi$  is **derivable**, then so is  $\Gamma''_0 \Rightarrow \varphi$  for any  $\Gamma''_0$  that contains the same **sentences** as  $\Gamma'_0$ . For instance, if  $\Gamma_0 = \{\psi, \chi\}$  then both  $\Gamma'_0 = \langle \psi, \psi, \chi \rangle$  and  $\Gamma''_0 = \langle \chi, \chi, \psi \rangle$  are sequences containing just the **sentences** in  $\Gamma_0$ . If a sequent containing one is **derivable**, so is the other, e.g.:

$$\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\frac{\psi, \psi, \chi \Rightarrow \varphi}{\psi, \chi \Rightarrow \varphi} \text{CL} \\
\frac{\psi, \chi \Rightarrow \varphi}{\chi, \psi \Rightarrow \varphi} \text{XL} \\
\frac{\chi, \psi \Rightarrow \varphi}{\chi, \chi, \psi \Rightarrow \varphi} \text{WL}
\end{array}$$

From now on we'll say that if  $\Gamma_0$  is a finite set of **sentences** then  $\Gamma_0 \Rightarrow \varphi$  is any sequent where the antecedent is a sequence of **sentences** in  $\Gamma_0$  and tacitly include contractions, exchanges, and weakenings if necessary.

**Definition seq.12** (Consistency). A set of sentences  $\Gamma$  is *inconsistent* iff there is a finite subset  $\Gamma_0 \subseteq \Gamma$  such that **LK** derives  $\Gamma_0 \Rightarrow$  . If  $\Gamma$  is not inconsistent, i.e., if for every finite  $\Gamma_0 \subseteq \Gamma$ , **LK** does not derive  $\Gamma_0 \Rightarrow$  , we say it is *consistent*.

**Proposition seq.13** (Reflexivity). *If  $\varphi \in \Gamma$ , then  $\Gamma \vdash \varphi$ .*

*fol:seq:ptn:  
prop:reflexivity*

*Proof.* The initial sequent  $\varphi \Rightarrow \varphi$  is **derivable**, and  $\{\varphi\} \subseteq \Gamma$ . □

**Proposition seq.14** (Monotony). *If  $\Gamma \subseteq \Delta$  and  $\Gamma \vdash \varphi$ , then  $\Delta \vdash \varphi$ .*

*fol:seq:ptn:  
prop:monotony*

*Proof.* Suppose  $\Gamma \vdash \varphi$ , i.e., there is a finite  $\Gamma_0 \subseteq \Gamma$  such that  $\Gamma_0 \Rightarrow \varphi$  is **derivable**. Since  $\Gamma \subseteq \Delta$ , then  $\Gamma_0$  is also a finite subset of  $\Delta$ . The **derivation** of  $\Gamma_0 \Rightarrow \varphi$  thus also shows  $\Delta \vdash \varphi$ . □

**Proposition seq.15** (Transitivity). *If  $\Gamma \vdash \varphi$  and  $\{\varphi\} \cup \Delta \vdash \psi$ , then  $\Gamma \cup \Delta \vdash \psi$ .*

*fol:seq:ptn:  
prop:transitivity*

*Proof.* If  $\Gamma \vdash \varphi$ , there is a finite  $\Gamma_0 \subseteq \Gamma$  and a **derivation**  $\pi_0$  of  $\Gamma_0 \Rightarrow \varphi$ . If  $\{\varphi\} \cup \Delta \vdash \psi$ , then for some finite subset  $\Delta_0 \subseteq \Delta$ , there is a **derivation**  $\pi_1$  of  $\varphi, \Delta_0 \Rightarrow \psi$ . Consider the following **derivation**:

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \pi_0 \\ \vdots \\ \vdots \\ \vdots \\ \Gamma_0 \Rightarrow \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \pi_1 \\ \vdots \\ \vdots \\ \vdots \\ \varphi, \Delta_0 \Rightarrow \psi \end{array}}{\Gamma_0, \Delta_0 \Rightarrow \psi} \text{Cut}$$

Since  $\Gamma_0 \cup \Delta_0 \subseteq \Gamma \cup \Delta$ , this shows  $\Gamma \cup \Delta \vdash \psi$ . □

Note that this means that in particular if  $\Gamma \vdash \varphi$  and  $\varphi \vdash \psi$ , then  $\Gamma \vdash \psi$ . It follows also that if  $\varphi_1, \dots, \varphi_n \vdash \psi$  and  $\Gamma \vdash \varphi_i$  for each  $i$ , then  $\Gamma \vdash \psi$ .

**Proposition seq.16.**  *$\Gamma$  is inconsistent iff  $\Gamma \vdash \varphi$  for every sentence  $\varphi$ .*

*fol:seq:ptn:  
prop:incons*

*Proof.* Exercise. □

**Problem seq.3.** Prove [Proposition seq.16](#)

**Proposition seq.17** (Compactness).

*fol:seq:ptn:  
prop:proves-compact*

1. If  $\Gamma \vdash \varphi$  then there is a finite subset  $\Gamma_0 \subseteq \Gamma$  such that  $\Gamma_0 \vdash \varphi$ .
2. If every finite subset of  $\Gamma$  is consistent, then  $\Gamma$  is consistent.

*Proof.* 1. If  $\Gamma \vdash \varphi$ , then there is a finite subset  $\Gamma_0 \subseteq \Gamma$  such that the sequent  $\Gamma_0 \Rightarrow \varphi$  has a **derivation**. Consequently,  $\Gamma_0 \vdash \varphi$ .

2. If  $\Gamma$  is inconsistent, there is a finite subset  $\Gamma_0 \subseteq \Gamma$  such that **LK derives**  $\Gamma_0 \Rightarrow \perp$ . But then  $\Gamma_0$  is a finite subset of  $\Gamma$  that is inconsistent. □

## seq.9 Derivability and Consistency

fol:seq:prv: sec We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:seq:prv: prop:provability-contr **Proposition seq.18.** *If  $\Gamma \vdash \varphi$  and  $\Gamma \cup \{\varphi\}$  is inconsistent, then  $\Gamma$  is inconsistent.*

*Proof.* There are finite  $\Gamma_0$  and  $\Gamma_1 \subseteq \Gamma$  such that **LK derives**  $\Gamma_0 \Rightarrow \varphi$  and  $\varphi, \Gamma_1 \Rightarrow \perp$ . Let the **LK-derivation** of  $\Gamma_0 \Rightarrow \varphi$  be  $\pi_0$  and the **LK-derivation** of  $\Gamma_1, \varphi \Rightarrow \perp$  be  $\pi_1$ . We can then **derive**

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \pi_0 \\ \vdots \\ \vdots \\ \Gamma_0 \Rightarrow \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ \pi_1 \\ \vdots \\ \vdots \\ \varphi, \Gamma_1 \Rightarrow \perp \end{array}}{\Gamma_0, \Gamma_1 \Rightarrow \perp} \text{Cut}$$

Since  $\Gamma_0 \subseteq \Gamma$  and  $\Gamma_1 \subseteq \Gamma$ ,  $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$ , hence  $\Gamma$  is inconsistent. □

fol:seq:prv: prop:prov-incons **Proposition seq.19.**  *$\Gamma \vdash \varphi$  iff  $\Gamma \cup \{\neg\varphi\}$  is inconsistent.*

*Proof.* First suppose  $\Gamma \vdash \varphi$ , i.e., there is a **derivation**  $\pi_0$  of  $\Gamma \Rightarrow \varphi$ . By adding a  $\neg$ L rule, we obtain a **derivation** of  $\neg\varphi, \Gamma \Rightarrow \perp$ , i.e.,  $\Gamma \cup \{\neg\varphi\}$  is inconsistent.

If  $\Gamma \cup \{\neg\varphi\}$  is inconsistent, there is a **derivation**  $\pi_1$  of  $\neg\varphi, \Gamma \Rightarrow \perp$ . The following is a **derivation** of  $\Gamma \Rightarrow \varphi$ :

$$\frac{\frac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi} \neg\text{R} \quad \begin{array}{c} \vdots \\ \vdots \\ \pi_1 \\ \vdots \\ \vdots \\ \neg\varphi, \Gamma \Rightarrow \perp \end{array}}{\Gamma \Rightarrow \varphi} \text{Cut}$$

□

**Problem seq.4.** Prove that  $\Gamma \vdash \neg\varphi$  iff  $\Gamma \cup \{\varphi\}$  is inconsistent.

fol:seq:prv: prop:explicit-inc **Proposition seq.20.** *If  $\Gamma \vdash \varphi$  and  $\neg\varphi \in \Gamma$ , then  $\Gamma$  is inconsistent.*

*Proof.* Suppose  $\Gamma \vdash \varphi$  and  $\neg\varphi \in \Gamma$ . Then there is a **derivation**  $\pi$  of a sequent  $\Gamma_0 \Rightarrow \varphi$ . The sequent  $\neg\varphi, \Gamma_0 \Rightarrow$  is also **derivable**:

$$\frac{\frac{\frac{\vdots \pi}{\Gamma_0 \Rightarrow \varphi}}{\neg\varphi, \Gamma_0 \Rightarrow} \text{XL} \quad \frac{\varphi \Rightarrow \varphi}{\varphi, \neg\varphi \Rightarrow} \text{XL}}{\Gamma, \neg\varphi \Rightarrow} \text{Cut}$$

Since  $\neg\varphi \in \Gamma$  and  $\Gamma_0 \subseteq \Gamma$ , this shows that  $\Gamma$  is inconsistent.  $\square$

**Proposition seq.21.** *If  $\Gamma \cup \{\varphi\}$  and  $\Gamma \cup \{\neg\varphi\}$  are both inconsistent, then  $\Gamma$  is inconsistent.* fol:seq:prv: prop:provability-exhaustive

*Proof.* There are finite sets  $\Gamma_0 \subseteq \Gamma$  and  $\Gamma_1 \subseteq \Gamma$  and **LK-derivations**  $\pi_0$  and  $\pi_1$  of  $\varphi, \Gamma_0 \Rightarrow$  and  $\neg\varphi, \Gamma_1 \Rightarrow$ , respectively. We can then **derive**

$$\frac{\frac{\frac{\vdots \pi_0}{\varphi, \Gamma_0 \Rightarrow}}{\Gamma_0 \Rightarrow \neg\varphi} \text{-R} \quad \frac{\frac{\vdots \pi_1}{\neg\varphi, \Gamma_1 \Rightarrow}}{\Gamma_0, \Gamma_1 \Rightarrow} \text{Cut}}{\Gamma, \neg\varphi \Rightarrow} \text{Cut}$$

Since  $\Gamma_0 \subseteq \Gamma$  and  $\Gamma_1 \subseteq \Gamma$ ,  $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$ . Hence  $\Gamma$  is inconsistent.  $\square$

## seq.10 Derivability and the Propositional Connectives

**Proposition seq.22.**

1. Both  $\varphi \wedge \psi \vdash \varphi$  and  $\varphi \wedge \psi \vdash \psi$ .
2.  $\varphi, \psi \vdash \varphi \wedge \psi$ .

*Proof.* 1. Both sequents  $\varphi \wedge \psi \Rightarrow \varphi$  and  $\varphi \wedge \psi \Rightarrow \psi$  are **derivable**:

$$\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi} \wedge\text{L} \quad \frac{\psi \Rightarrow \psi}{\varphi \wedge \psi \Rightarrow \psi} \wedge\text{L}$$

2. Here is a **derivation** of the sequent  $\varphi, \psi \Rightarrow \varphi \wedge \psi$ :

$$\frac{\varphi \Rightarrow \varphi \quad \psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \varphi \wedge \psi} \wedge\text{R}$$

$\square$

**Proposition seq.23.**

1.  $\varphi \vee \psi, \neg\varphi, \neg\psi$  is inconsistent.

fol:seq:ppr: prop:provability-lor

2. Both  $\varphi \vdash \varphi \vee \psi$  and  $\psi \vdash \varphi \vee \psi$ .

*Proof.* 1. We give a **derivation** of the sequent  $\varphi \vee \psi, \neg\varphi, \neg\psi \Rightarrow$ :

$$\frac{\frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg\text{L}}{\varphi, \neg\varphi, \neg\psi \Rightarrow} \quad \frac{\frac{\frac{\psi \Rightarrow \psi}{\neg\psi, \psi \Rightarrow} \neg\text{L}}{\psi, \neg\varphi, \neg\psi \Rightarrow} \neg\text{L}}{\varphi \vee \psi, \neg\varphi, \neg\psi \Rightarrow} \vee\text{L}$$

(Recall that double inference lines indicate several weakening, contraction, and exchange inferences.)

2. Both sequents  $\varphi \Rightarrow \varphi \vee \psi$  and  $\psi \Rightarrow \varphi \vee \psi$  have **derivations**:

$$\frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee \psi} \vee\text{R} \quad \frac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi \vee \psi} \vee\text{R}$$

□

*fol:seq:ppr:* **Proposition seq.24.**

*prop:provability-lif*

*fol:seq:ppr:*

*prop:provability-lif-left*

1.  $\varphi, \varphi \rightarrow \psi \vdash \psi$ .

*fol:seq:ppr:*

*prop:provability-lif-right*

2. Both  $\neg\varphi \vdash \varphi \rightarrow \psi$  and  $\psi \vdash \varphi \rightarrow \psi$ .

*Proof.* 1. The sequent  $\varphi \rightarrow \psi, \varphi \Rightarrow \psi$  is **derivable**:

$$\frac{\varphi \Rightarrow \varphi \quad \psi \Rightarrow \psi}{\varphi \rightarrow \psi, \varphi \Rightarrow \psi} \rightarrow\text{L}$$

2. Both sequents  $\neg\varphi \Rightarrow \varphi \rightarrow \psi$  and  $\psi \Rightarrow \varphi \rightarrow \psi$  are **derivable**:

$$\frac{\frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg\text{L}}{\varphi, \neg\varphi \Rightarrow} \text{XL}}{\varphi, \neg\varphi \Rightarrow \psi} \text{WR} \quad \frac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} \text{WL}}{\neg\varphi \Rightarrow \varphi \rightarrow \psi} \rightarrow\text{R} \quad \frac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi \rightarrow \psi} \rightarrow\text{R}$$

□

## seq.11 Derivability and the Quantifiers

fol:seq:qpr:  
sec

**Theorem seq.25.** *If  $c$  is a constant not occurring in  $\Gamma$  or  $\varphi(x)$  and  $\Gamma \vdash \varphi(c)$ , then  $\Gamma \vdash \forall x \varphi(x)$ .*

fol:seq:qpr:  
thm:strong-generalization

*Proof.* Let  $\pi_0$  be an **LK-derivation** of  $\Gamma_0 \Rightarrow \varphi(c)$  for some finite  $\Gamma_0 \subseteq \Gamma$ . By adding a  $\forall R$  inference, we obtain a proof of  $\Gamma_0 \Rightarrow \forall x \varphi(x)$ , since  $c$  does not occur in  $\Gamma$  or  $\varphi(x)$  and thus the eigenvariable condition is satisfied.  $\square$

**Proposition seq.26.**

fol:seq:qpr:  
prop:provability-quantifiers

1.  $\varphi(t) \vdash \exists x \varphi(x)$ .
2.  $\forall x \varphi(x) \vdash \varphi(t)$ .

*Proof.* 1. The sequent  $\varphi(t) \Rightarrow \exists x \varphi(x)$  is **derivable**:

$$\frac{\varphi(t) \Rightarrow \varphi(t)}{\varphi(t) \Rightarrow \exists x \varphi(x)} \exists R$$

2. The sequent  $\forall x \varphi(x) \Rightarrow \varphi(t)$  is **derivable**:

$$\frac{\varphi(t) \Rightarrow \varphi(t)}{\forall x \varphi(x) \Rightarrow \varphi(t)} \forall L$$

$\square$

## seq.12 Soundness

explanation

A **derivation** system, such as the sequent calculus, is *sound* if it cannot **derive** things that do not actually hold. Soundness is thus a kind of guaranteed safety property for **derivation** systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

fol:seq:sou:  
sec

1. every **derivable**  $\varphi$  is valid;
2. if a **sentence** is **derivable** from some others, it is also a consequence of them;
3. if a set of **sentences** is inconsistent, it is unsatisfiable.

These are important properties of a **derivation** system. If any of them do not hold, the **derivation** system is deficient—it would **derive** too much. Consequently, establishing the soundness of a **derivation** system is of the utmost importance.

Because all these proof-theoretic properties are defined via **derivability** in the sequent calculus of certain sequents, proving (1)–(3) above requires proving

something about the semantic properties of **derivable** sequents. We will first define what it means for a sequent to be *valid*, and then show that every **derivable** sequent is valid. (1)–(3) then follow as corollaries from this result.

**Definition seq.27.** A **structure**  $\mathfrak{M}$  *satisfies* a sequent  $\Gamma \Rightarrow \Delta$  iff either  $\mathfrak{M} \not\models \varphi$  for some  $\varphi \in \Gamma$  or  $\mathfrak{M} \models \varphi$  for some  $\varphi \in \Delta$ .

A sequent is *valid* iff every **structure**  $\mathfrak{M}$  satisfies it.

*fol:seq:sou:*  
*thm:sequent-soundness*

**Theorem seq.28** (Soundness). *If LK derives  $\Theta \Rightarrow \Xi$ , then  $\Theta \Rightarrow \Xi$  is valid.*

*Proof.* Let  $\pi$  be a **derivation** of  $\Theta \Rightarrow \Xi$ . We proceed by induction on the number of inferences  $n$  in  $\pi$ .

If the number of inferences is 0, then  $\pi$  consists only of an initial sequent. Every initial sequent  $\varphi \Rightarrow \varphi$  is obviously valid, since for every  $\mathfrak{M}$ , either  $\mathfrak{M} \not\models \varphi$  or  $\mathfrak{M} \models \varphi$ .

If the number of inferences is greater than 0, we distinguish cases according to the type of the lowermost inference. By induction hypothesis, we can assume that the premises of that inference are valid, since the number of inferences in the proof of any premise is smaller than  $n$ .

First, we consider the possible inferences with only one premise.

1. The last inference is a weakening. Then  $\Theta \Rightarrow \Xi$  is either  $A, \Gamma \Rightarrow \Delta$  (if the last inference is WL) or  $\Gamma \Rightarrow \Delta, \varphi$  (if it's WR), and the **derivation** ends in one of

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta \end{array}}{\varphi, \Gamma \Rightarrow \Delta} \text{WL} \qquad \frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta \end{array}}{\Gamma \Rightarrow \Delta, \varphi} \text{WR}$$

By induction hypothesis,  $\Gamma \Rightarrow \Delta$  is valid, i.e., for every **structure**  $\mathfrak{M}$ , either there is some  $\chi \in \Gamma$  such that  $\mathfrak{M} \not\models \chi$  or there is some  $\chi \in \Delta$  such that  $\mathfrak{M} \models \chi$ .

If  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , then  $\chi \in \Theta$  as well since  $\Theta = \varphi, \Gamma$ , and so  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Theta$ . Similarly, if  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ , as  $\chi \in \Xi$ ,  $\mathfrak{M} \models \chi$  for some  $\chi \in \Xi$ . Consequently,  $\Theta \Rightarrow \Xi$  is valid.

2. The last inference is  $\neg$ L: Then the premise of the last inference is  $\Gamma \Rightarrow \Delta, \varphi$  and the conclusion is  $\neg\varphi, \Gamma \Rightarrow \Delta$ , i.e., the **derivation** ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array}}{\neg\varphi, \Gamma \Rightarrow \Delta} \neg\text{L}$$

and  $\Theta = \neg\varphi, \Gamma$  while  $\Xi = \Delta$ .

The induction hypothesis tells us that  $\Gamma \Rightarrow \Delta, \varphi$  is valid, i.e., for every  $\mathfrak{M}$ , either (a) for some  $\chi \in \Gamma$ ,  $\mathfrak{M} \not\models \chi$ , or (b) for some  $\chi \in \Delta$ ,  $\mathfrak{M} \models \chi$ , or (c)  $\mathfrak{M} \models \varphi$ . We want to show that  $\Theta \Rightarrow \Xi$  is also valid. Let  $\mathfrak{M}$  be a structure. If (a) holds, then there is  $\chi \in \Gamma$  so that  $\mathfrak{M} \not\models \varphi$ , but  $\varphi \in \Theta$  as well. If (b) holds, there is  $\chi \in \Delta$  such that  $\mathfrak{M} \models \chi$ , but  $\chi \in \Xi$  as well. Finally, if  $\mathfrak{M} \models \varphi$ , then  $\mathfrak{M} \not\models \neg\varphi$ . Since  $\neg\varphi \in \Theta$ , there is  $\chi \in \Theta$  such that  $\mathfrak{M} \not\models \chi$ . Consequently,  $\Theta \Rightarrow \Xi$  is valid.

3. The last inference is  $\neg$ R: Exercise.
4. The last inference is  $\wedge$ L: There are two variants:  $\varphi \wedge \psi$  may be inferred on the left from  $\varphi$  or from  $\psi$  on the left side of the premise. In the first case, the  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \varphi, \Gamma \Rightarrow \Delta \end{array}}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\text{L}$$

and  $\Theta = \varphi \wedge \psi, \Gamma$  while  $\Xi = \Delta$ . Consider a structure  $\mathfrak{M}$ . Since by induction hypothesis,  $\varphi, \Gamma \Rightarrow \Delta$  is valid, (a)  $\mathfrak{M} \not\models \varphi$ , (b)  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , or (c)  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ . In case (a),  $\mathfrak{M} \not\models \varphi \wedge \psi$ , so there is  $\chi \in \Theta$  (namely,  $\varphi \wedge \psi$ ) such that  $\mathfrak{M} \not\models \chi$ . In case (b), there is  $\chi \in \Gamma$  such that  $\mathfrak{M} \not\models \chi$ , and  $\chi \in \Theta$  as well. In case (c), there is  $\chi \in \Delta$  such that  $\mathfrak{M} \models \chi$ , and  $\chi \in \Xi$  as well since  $\Xi = \Delta$ . So in each case,  $\mathfrak{M}$  satisfies  $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$ . Since  $\mathfrak{M}$  was arbitrary,  $\Gamma \Rightarrow \Delta$  is valid. The case where  $\varphi \wedge \psi$  is inferred from  $\psi$  is handled the same, changing  $\varphi$  to  $\psi$ .

5. The last inference is  $\vee$ R: There are two variants:  $\varphi \vee \psi$  may be inferred on the right from  $\varphi$  or from  $\psi$  on the right side of the premise. In the first case,  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array}}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \vee\text{R}$$

Now  $\Theta = \Gamma$  and  $\Xi = \Delta, \varphi \vee \psi$ . Consider a structure  $\mathfrak{M}$ . Since  $\Gamma \Rightarrow \Delta, \varphi$  is valid, (a)  $\mathfrak{M} \models \varphi$ , (b)  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , or (c)  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ . In case (a),  $\mathfrak{M} \models \varphi \vee \psi$ . In case (b), there is  $\chi \in \Gamma$  such that  $\mathfrak{M} \not\models \chi$ . In case (c), there is  $\chi \in \Delta$  such that  $\mathfrak{M} \models \chi$ . So in each case,  $\mathfrak{M}$  satisfies  $\Gamma \Rightarrow \Delta, \varphi \vee \psi$ , i.e.,  $\Theta \Rightarrow \Xi$ . Since  $\mathfrak{M}$  was arbitrary,  $\Theta \Rightarrow \Xi$  is valid. The case where  $\varphi \vee \psi$  is inferred from  $\psi$  is handled the same, changing  $\varphi$  to  $\psi$ .

6. The last inference is  $\rightarrow R$ : Then  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \varphi, \Gamma \Rightarrow \Delta, \varphi \end{array}}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \rightarrow R$$

Again, the induction hypothesis says that the premise is valid; we want to show that the conclusion is valid as well. Let  $\mathfrak{M}$  be arbitrary. Since  $\varphi, \Gamma \Rightarrow \Delta, \psi$  is valid, at least one of the following cases obtains: (a)  $\mathfrak{M} \not\models \varphi$ , (b)  $\mathfrak{M} \models \psi$ , (c)  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , or (d)  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ . In cases (a) and (b),  $\mathfrak{M} \models \varphi \rightarrow \psi$  and so there is a  $\chi \in \Delta, \varphi \rightarrow \psi$  such that  $\mathfrak{M} \models \chi$ . In case (c), for some  $\chi \in \Gamma$ ,  $\mathfrak{M} \not\models \chi$ . In case (d), for some  $\chi \in \Delta$ ,  $\mathfrak{M} \models \chi$ . In each case,  $\mathfrak{M}$  satisfies  $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$ . Since  $\mathfrak{M}$  was arbitrary,  $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$  is valid.

7. The last inference is  $\forall L$ : Then there is a **formula**  $\varphi(x)$  and a closed term  $t$  such that  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \varphi(t), \Gamma \Rightarrow \Delta \end{array}}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \forall L$$

We want to show that the conclusion  $\forall x \varphi(x), \Gamma \Rightarrow \Delta$  is valid. Consider a **structure**  $\mathfrak{M}$ . Since the premise  $\varphi(t), \Gamma \Rightarrow \Delta$  is valid, (a)  $\mathfrak{M} \not\models \varphi(t)$ , (b)  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , or (c)  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ . In case (a), by ??, if  $\mathfrak{M} \models \forall x \varphi(x)$ , then  $\mathfrak{M} \models \varphi(t)$ . Since  $\mathfrak{M} \not\models \varphi(t)$ ,  $\mathfrak{M} \not\models \forall x \varphi(x)$ . In case (b) and (c),  $\mathfrak{M}$  also satisfies  $\forall x \varphi(x), \Gamma \Rightarrow \Delta$ . Since  $\mathfrak{M}$  was arbitrary,  $\forall x \varphi(x), \Gamma \Rightarrow \Delta$  is valid.

8. The last inference is  $\exists R$ : Exercise.

9. The last inference is  $\forall R$ : Then there is a **formula**  $\varphi(x)$  and a **constant symbol**  $a$  such that  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta, \varphi(a) \end{array}}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} \forall R$$

where the eigenvariable condition is satisfied, i.e.,  $a$  does not occur in  $\varphi(x)$ ,  $\Gamma$ , or  $\Delta$ . By induction hypothesis, the premise of the last inference is valid. We have to show that the conclusion is valid as well, i.e., that

for any **structure**  $\mathfrak{M}$ , (a)  $\mathfrak{M} \models \forall x \varphi(x)$ , (b)  $\mathfrak{M} \not\models \chi$  for some  $\chi \in \Gamma$ , or (c)  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$ .

Suppose  $\mathfrak{M}$  is an arbitrary **structure**. If (b) or (c) holds, we are done, so suppose neither holds: for all  $\chi \in \Gamma$ ,  $\mathfrak{M} \models \chi$ , and for all  $\chi \in \Delta$ ,  $\mathfrak{M} \not\models \chi$ . We have to show that (a) holds, i.e.,  $\mathfrak{M} \models \forall x \varphi(x)$ . By ??, it suffices to show that  $\mathfrak{M}, s \models \varphi(x)$  for all variable assignments  $s$ . So let  $s$  be an arbitrary variable assignment. Consider the structure  $\mathfrak{M}'$  which is just like  $\mathfrak{M}$  except  $a^{\mathfrak{M}'} = s(x)$ . By ??, for any  $\chi \in \Gamma$ ,  $\mathfrak{M}' \models \chi$  since  $a$  does not occur in  $\Gamma$ , and for any  $\chi \in \Delta$ ,  $\mathfrak{M}' \not\models \chi$ . But the premise is valid, so  $\mathfrak{M}' \models \varphi(a)$ . By ??,  $\mathfrak{M}', s \models \varphi(a)$ , since  $\varphi(a)$  is a sentence. Now  $s \sim_x s$  with  $s(x) = \text{Val}_s^{\mathfrak{M}'}(a)$ , since we've defined  $\mathfrak{M}'$  in just this way. So ?? applies, and we get  $\mathfrak{M}', s \models \varphi(x)$ . Since  $a$  does not occur in  $\varphi(x)$ , by ??,  $\mathfrak{M}, s \models \varphi(x)$ . Since  $s$  was arbitrary, we've completed the proof that  $\mathfrak{M}, s \models \varphi(x)$  for all variable assignments.

10. The last inference is  $\exists$ L: Exercise.

Now let's consider the possible inferences with two premises.

1. The last inference is a cut: then  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \varphi, \Pi \Rightarrow \Lambda \end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \text{Cut}$$

Let  $\mathfrak{M}$  be a **structure**. By induction hypothesis, the premises are valid, so  $\mathfrak{M}$  satisfies both premises. We distinguish two cases: (a)  $\mathfrak{M} \not\models \varphi$  and (b)  $\mathfrak{M} \models \varphi$ . In case (a), in order for  $\mathfrak{M}$  to satisfy the left premise, it must satisfy  $\Gamma \Rightarrow \Delta$ . But then it also satisfies the conclusion. In case (b), in order for  $\mathfrak{M}$  to satisfy the right premise, it must satisfy  $\Pi \setminus \Lambda$ . Again,  $\mathfrak{M}$  satisfies the conclusion.

2. The last inference is  $\wedge$ R. Then  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta, \psi \end{array}}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge R$$

Consider a **structure**  $\mathfrak{M}$ . If  $\mathfrak{M}$  satisfies  $\Gamma \Rightarrow \Delta$ , we are done. So suppose it doesn't. Since  $\Gamma \Rightarrow \Delta, \varphi$  is valid by induction hypothesis,  $\mathfrak{M} \models \varphi$ . Similarly, since  $\Gamma \Rightarrow \Delta, \psi$  is valid,  $\mathfrak{M} \models \psi$ . But then  $\mathfrak{M} \models \varphi \wedge \psi$ .

3. The last inference is  $\vee$ L: Exercise.

4. The last inference is  $\rightarrow$ L. Then  $\pi$  ends in

$$\frac{\begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array} \quad \begin{array}{c} \vdots \\ \psi, \Pi \Rightarrow \Lambda \end{array}}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \rightarrow L$$

Again, consider a **structure**  $\mathfrak{M}$  and suppose  $\mathfrak{M}$  doesn't satisfy  $\Gamma, \Pi \Rightarrow \Delta, \Lambda$ . We have to show that  $\mathfrak{M} \not\models \varphi \rightarrow \psi$ . If  $\mathfrak{M}$  doesn't satisfy  $\Gamma, \Pi \Rightarrow \Delta, \Lambda$ , it satisfies neither  $\Gamma \Rightarrow \Delta$  nor  $\Pi \Rightarrow \Lambda$ . Since,  $\Gamma \Rightarrow \Delta, \varphi$  is valid, we have  $\mathfrak{M} \models \varphi$ . Since  $\psi, \Pi \Rightarrow \Lambda$  is valid, we have  $\mathfrak{M} \not\models \psi$ . But then  $\mathfrak{M} \not\models \varphi \rightarrow \psi$ , which is what we wanted to show.

□

**Problem seq.5.** Complete the proof of [Theorem seq.28](#).

*fol:seq:sou:  
cor:weak-soundness*

**Corollary seq.29.** *If  $\vdash \varphi$  then  $\varphi$  is valid.*

*fol:seq:sou:  
cor:entailment-soundness*

**Corollary seq.30.** *If  $\Gamma \vdash \varphi$  then  $\Gamma \models \varphi$ .*

*Proof.* If  $\Gamma \vdash \varphi$  then for some finite subset  $\Gamma_0 \subseteq \Gamma$ , there is a **derivation** of  $\Gamma_0 \Rightarrow \varphi$ . By [Theorem seq.28](#), every **structure**  $\mathfrak{M}$  either makes some  $\psi \in \Gamma_0$  false or makes  $\varphi$  true. Hence, if  $\mathfrak{M} \models \Gamma$  then also  $\mathfrak{M} \models \varphi$ . □

*fol:seq:sou:  
cor:consistency-soundness*

**Corollary seq.31.** *If  $\Gamma$  is satisfiable, then it is consistent.*

*Proof.* We prove the contrapositive. Suppose that  $\Gamma$  is not consistent. Then there is a finite  $\Gamma_0 \subseteq \Gamma$  and a **derivation** of  $\Gamma_0 \Rightarrow \perp$ . By [Theorem seq.28](#),  $\Gamma_0 \Rightarrow \perp$  is valid. In other words, for every **structure**  $\mathfrak{M}$ , there is  $\chi \in \Gamma_0$  so that  $\mathfrak{M} \not\models \chi$ , and since  $\Gamma_0 \subseteq \Gamma$ , that  $\chi$  is also in  $\Gamma$ . Thus, no  $\mathfrak{M}$  satisfies  $\Gamma$ , and  $\Gamma$  is not satisfiable. □

## seq.13 Derivations with Identity predicate

*fol:seq:ide:  
sec*

**Derivations** with **identity predicate** require additional initial sequents and inference rules.

**Definition seq.32** (Initial sequents for =). If  $t$  is a closed term, then  $\Rightarrow t = t$  is an initial sequent.

The rules for = are ( $t_1$  and  $t_2$  are closed terms):

$$\boxed{\frac{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)}{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)} = \quad \frac{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)}{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)} =}$$

**Example seq.33.** If  $s$  and  $t$  are closed terms, then  $s = t, \varphi(s) \vdash \varphi(t)$ :

$$\frac{\frac{\varphi(s) \Rightarrow \varphi(s)}{s = t, \varphi(s) \Rightarrow \varphi(s)} \text{WL}}{s = t, \varphi(s) \Rightarrow \varphi(t)} =$$

This may be familiar as the principle of substitutability of identicals, or Leibniz' Law.

**LK** proves that  $=$  is symmetric and transitive:

$$\frac{\frac{\Rightarrow t_1 = t_1}{t_1 = t_2 \Rightarrow t_1 = t_1} \text{WL}}{t_1 = t_2 \Rightarrow t_2 = t_1} = \quad \frac{\frac{t_1 = t_2 \Rightarrow t_1 = t_2}{t_2 = t_3, t_1 = t_2 \Rightarrow t_1 = t_2} \text{WL}}{t_1 = t_2, t_2 = t_3 \Rightarrow t_1 = t_3} \text{XL}$$

In the proof on the left, the formula  $x = t_1$  is our  $\varphi(x)$ . On the right, we take  $\varphi(x)$  to be  $t_1 = x$ .

**Problem seq.6.** Give derivations of the following sequents:

1.  $\Rightarrow \forall x \forall y ((x = y \wedge \varphi(x)) \rightarrow \varphi(y))$
2.  $\exists x \varphi(x) \wedge \forall y \forall z ((\varphi(y) \wedge \varphi(z)) \rightarrow y = z) \Rightarrow \exists x (\varphi(x) \wedge \forall y (\varphi(y) \rightarrow y = x))$

## seq.14 Soundness with Identity predicate

fol:seq:sid:  
sec

**Proposition seq.34.** **LK** with initial sequents and rules for identity is sound.

*Proof.* Initial sequents of the form  $\Rightarrow t = t$  are valid, since for every structure  $\mathfrak{M}$ ,  $\mathfrak{M} \models t = t$ . (Note that we assume the term  $t$  to be closed, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a derivation is  $=$ . Then the premise is  $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)$  and the conclusion is  $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)$ . Consider a structure  $\mathfrak{M}$ . We need to show that the conclusion is valid, i.e., if  $\mathfrak{M} \models t_1 = t_2$  and  $\mathfrak{M} \models \Gamma$ , then either  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$  or  $\mathfrak{M} \models \varphi(t_2)$ .

By induction hypothesis, the premise is valid. This means that if  $\mathfrak{M} \models t_1 = t_2$  and  $\mathfrak{M} \models \Gamma$  either (a) for some  $\chi \in \Delta$ ,  $\mathfrak{M} \models \chi$  or (b)  $\mathfrak{M} \models \varphi(t_1)$ . In case (a) we are done. Consider case (b). Let  $s$  be a variable assignment with  $s(x) = \text{Val}^{\mathfrak{M}}(t_1)$ . By ??,  $\mathfrak{M}, s \models \varphi(t_1)$ . Since  $s \sim_x s$ , by ??,  $\mathfrak{M}, s \models \varphi(x)$ . since  $\mathfrak{M} \models t_1 = t_2$ , we have  $\text{Val}^{\mathfrak{M}}(t_1) = \text{Val}^{\mathfrak{M}}(t_2)$ , and hence  $s(x) = \text{Val}^{\mathfrak{M}}(t_2)$ . By applying ?? again, we also have  $\mathfrak{M}, s \models \varphi(t_2)$ . By ??,  $\mathfrak{M} \models \varphi(t_2)$ .  $\square$

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