seq.1 Quantifier Rules

fol:seq:qrl:sec

Rules for $\forall$

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\ \forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\ \forall\mathrm{R}$$

In $\forall$L, t is a closed term (i.e., one without variables). In $\forall$R, a is a constant symbol which must not occur anywhere in the lower sequent of the $\forall$R rule. We call a the *eigenvariable* of the $\forall$R inference.[1]

Rules for $\exists$

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x\, \varphi(x), \Gamma \Rightarrow \Delta}\ \exists\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x)}\ \exists\mathrm{R}$$

Again, t is a closed term, and a is a constant symbol which does not occur in the lower sequent of the $\exists$L rule. We call a the *eigenvariable* of the $\exists$L inference.

The condition that an eigenvariable not occur in the lower sequent of the $\forall$R or $\exists$L inference is called the *eigenvariable condition*.

Recall the convention that when φ is a formula with the variable x free, we indicate this by writing $\varphi(x)$. In the same context, $\varphi(t)$ then is short for $\varphi[t/x]$. So we could also write the $\exists$R rule as: explanation

$$\frac{\Gamma \Rightarrow \Delta, \varphi[t/x]}{\Gamma \Rightarrow \Delta, \exists x\, \varphi}\ \exists\mathrm{R}$$

Note that t may already occur in φ, e.g., φ might be $P(t, x)$. Thus, inferring $\Gamma \Rightarrow \Delta, \exists x\, P(t, x)$ from $\Gamma \Rightarrow \Delta, P(t, t)$ is a correct application of $\exists$R—you may "replace" one or more, and not necessarily all, occurrences of t in the premise by the bound variable x. However, the eigenvariable conditions in $\forall$R and $\exists$L require that the constant symbol a does not occur in φ. So, you cannot correctly infer $\Gamma \Rightarrow \Delta, \forall x\, P(a, x)$ from $\Gamma \Rightarrow \Delta, P(a, a)$ using $\forall$R.

In $\exists$R and $\forall$L there are no restrictions on the term t. On the other hand, in the $\exists$L and $\forall$R rules, the eigenvariable condition requires that the constant symbol a does not occur anywhere outside of $\varphi(a)$ in the upper sequent. It is necessary to ensure that the system is sound, i.e., only derives sequents that are valid. Without this condition, the following would be allowed: explanation

[1] We use the term "eigenvariable" even though a in the above rule is a constant symbol. This has historical reasons.

$$\dfrac{\dfrac{\varphi(a) \;\Rightarrow\; \varphi(a)}{\exists x\,\varphi(x) \;\Rightarrow\; \varphi(a)}\,{*}\exists\mathrm{L}}{\exists x\,\varphi(x) \;\Rightarrow\; \forall x\,\varphi(x)}\,\forall\mathrm{R} \qquad \dfrac{\dfrac{\varphi(a) \;\Rightarrow\; \varphi(a)}{\varphi(a) \;\Rightarrow\; \forall x\,\varphi(x)}\,{*}\forall\mathrm{R}}{\exists x\,\varphi(x) \;\Rightarrow\; \forall x\,\varphi(x)}\,\exists\mathrm{L}$$

However, $\exists x\,\varphi(x) \Rightarrow \forall x\,\varphi(x)$ is not valid.

Photo Credits

Bibliography