seq.1  Quantifier Rules

Rules for ∀

\[
\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \forall L
\]
\[
\frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} \forall R
\]

In ∀L, \( t \) is a closed term (i.e., one without variables). In ∀R, \( a \) is a constant symbol which must not occur anywhere in the lower sequent of the ∀R rule. We call \( a \) the eigenvariable of the ∀R inference.

Rules for ∃

\[
\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x \varphi(x), \Gamma \Rightarrow \Delta} \exists L
\]
\[
\frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)} \exists R
\]

Again, \( t \) is a closed term, and \( a \) is a constant symbol which does not occur in the lower sequent of the ∃L rule. We call \( a \) the eigenvariable of the ∃L inference.

The condition that an eigenvariable not occur in the lower sequent of the ∀R or ∃L inference is called the eigenvariable condition.

We use the term “eigenvariable” even though \( a \) in the above rules is a constant symbol. This has historical reasons.

In ∃R and ∀L there are no restrictions on the term \( t \). On the other hand, in the ∃L and ∀R rules, the eigenvariable condition requires that the constant symbol \( a \) does not occur anywhere outside of \( \varphi(a) \) in the upper sequent. It is necessary to ensure that the system is sound, i.e., only derives sequents that are valid. Without this condition, the following would be allowed:

\[
\frac{\varphi(a) \Rightarrow \varphi(a)}{\exists x \varphi(x) \Rightarrow \varphi(a)} * \exists L
\]
\[
\frac{\varphi(a) \Rightarrow \varphi(a)}{\varphi(a) \Rightarrow \forall x \varphi(x)} * \forall R
\]
\[
\frac{\varphi(a) \Rightarrow \varphi(a)}{\forall x \varphi(x) \Rightarrow \varphi(a)} \forall L
\]
\[
\frac{\varphi(a) \Rightarrow \varphi(a)}{\exists x \varphi(x) \Rightarrow \forall x \varphi(x)} \exists L
\]

However, \( \exists x \varphi(x) \Rightarrow \forall x \varphi(x) \) is not valid.

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Bibliography