seq.1 Quantifier Rules

fol:seq:qrl:sec

Rules for $\forall$

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\ \forall\text{L} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\ \forall\text{R}$$

In $\forall$L, t is a closed term (i.e., one without variables). In $\forall$R, a is a constant symbol which must not occur anywhere in the lower sequent of the $\forall$R rule. We call a the *eigenvariable* of the $\forall$R inference.

Rules for $\exists$

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x\, \varphi(x), \Gamma \Rightarrow \Delta}\ \exists\text{L} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x)}\ \exists\text{R}$$

Again, t is a closed term, and a is a constant symbol which does not occur in the lower sequent of the $\exists$L rule. We call a the *eigenvariable* of the $\exists$L inference.

The condition that an eigenvariable not occur in the lower sequent of the $\forall$R or $\exists$L inference is called the *eigenvariable condition*.

We use the term "eigenvariable" even though a in the above rules is a constant symbol. This has historical reasons. explanation

In $\exists$R and $\forall$L there are no restrictions on the term t. On the other hand, in the $\exists$L and $\forall$R rules, the eigenvariable condition requires that the constant symbol a does not occur anywhere outside of $\varphi(a)$ in the upper sequent. It is necessary to ensure that the system is sound, i.e., only derives sequents that are valid. Without this condition, the following would be allowed:

$$\frac{\dfrac{\varphi(a) \Rightarrow \varphi(a)}{\exists x\, \varphi(x) \Rightarrow \varphi(a)}\ {*}\exists\text{L}}{\exists x\, \varphi(x) \Rightarrow \forall x\, \varphi(x)}\ \forall\text{R} \qquad\qquad \frac{\dfrac{\varphi(a) \Rightarrow \varphi(a)}{\varphi(a) \Rightarrow \forall x\, \varphi(x)}\ {*}\forall\text{R}}{\exists x\, \varphi(x) \Rightarrow \forall x\, \varphi(x)}\ \exists\text{L}$$

However, $\exists x\, \varphi(x) \Rightarrow \forall x\, \varphi(x)$ is not valid.

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Bibliography