seq.1 Quantifier Rules

Rules for $\forall$

$$
\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \quad \forall L
$$

$$
\frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} \quad \forall R
$$

In $\forall L$, $t$ is a closed term (i.e., one without variables). In $\forall R$, $a$ is a constant symbol which must not occur anywhere in the lower sequent of the $\forall R$ rule. We call $a$ the eigenvariable of the $\forall R$ inference.  

Rules for $\exists$

$$
\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x \varphi(x), \Gamma \Rightarrow \Delta} \quad \exists L
$$

$$
\frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)} \quad \exists R
$$

Again, $t$ is a closed term, and $a$ is a constant symbol which does not occur in the lower sequent of the $\exists L$ rule. We call $a$ the eigenvariable of the $\exists L$ inference.

The condition that an eigenvariable not occur in the lower sequent of the $\forall R$ or $\exists L$ inference is called the eigenvariable condition.

Recall the convention that when $\varphi$ is a formula with the variable $x$ free, we indicate this by writing $\varphi(x)$. In the same context, $\varphi(t)$ then is short for $\varphi[t/x]$. So we could also write the $\exists R$ rule as:

$$
\frac{\Gamma \Rightarrow \Delta, \varphi[t/x]}{\Gamma \Rightarrow \Delta, \exists x \varphi} \quad \exists R
$$

Note that $t$ may already occur in $\varphi$, e.g., $\varphi$ might be $P(t, x)$. Thus, inferring $\Gamma \Rightarrow \Delta, \exists x P(t, x)$ from $\Gamma \Rightarrow \Delta, P(t, t)$ is a correct application of $\exists R$—you may “replace” one or more, and not necessarily all, occurrences of $t$ in the premise by the bound variable $x$. However, the eigenvariable conditions in $\forall R$ and $\exists L$ require that the constant symbol $a$ does not occur in $\varphi$. So, you cannot correctly infer $\Gamma \Rightarrow \Delta, \forall x P(a, x)$ from $\Gamma \Rightarrow \Delta, P(a, a)$ using $\forall R$.

In $\exists R$ and $\forall L$ there are no restrictions on the term $t$. On the other hand, in the $\exists L$ and $\forall R$ rules, the eigenvariable condition requires that the constant symbol $a$ does not occur anywhere outside of $\varphi(a)$ in the upper sequent. It is necessary to ensure that the system is sound, i.e., only derives sequents that are valid. Without this condition, the following would be allowed:

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1We use the term “eigenvariable” even though $a$ in the above rule is a constant symbol. This has historical reasons.
However, $\exists x \varphi(x) \Rightarrow \forall x \varphi(x)$ is not valid.

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Bibliography