seq.1 Derivability and the Quantifiers

The completeness theorem also requires that the sequent calculus rules yield the facts about \( \vdash \) established in this section.

**Theorem seq.1.** If \( c \) is a constant not occurring in \( \Gamma \) or \( \varphi(x) \) and \( \Gamma \vdash \varphi(c) \), then \( \Gamma \vdash \forall x \varphi(x) \).

**Proof.** Let \( \pi_0 \) be an LK-derivation of \( \Gamma_0 \Rightarrow \varphi(c) \) for some finite \( \Gamma_0 \subseteq \Gamma \). By adding a \( \forall R \) inference, we obtain a derivation of \( \Gamma_0 \Rightarrow \forall x \varphi(x) \), since \( c \) does not occur in \( \Gamma \) or \( \varphi(x) \) and thus the eigenvariable condition is satisfied. \( \square \)

**Proposition seq.2.**

1. \( \varphi(t) \vdash \exists x \varphi(x) \).
2. \( \forall x \varphi(x) \vdash \varphi(t) \).

**Proof.**

1. The sequent \( \varphi(t) \Rightarrow \exists x \varphi(x) \) is derivable:

\[
\frac{\varphi(t) \Rightarrow \varphi(t)}{\varphi(t) \Rightarrow \exists x \varphi(x)} \exists R
\]

2. The sequent \( \forall x \varphi(x) \Rightarrow \varphi(t) \) is derivable:

\[
\frac{\varphi(t) \Rightarrow \varphi(t)}{\forall x \varphi(x) \Rightarrow \varphi(t)} \forall L
\]

\( \square \)

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Bibliography