seq.1 Derivability and the Propositional Connectives

fol:seq:ppr: sec

We establish that the derivability relation $\vdash$ of the sequent calculus is strong enough to establish some basic facts involving the propositional connectives, such as that $\varphi \land \psi \vdash \varphi$ and $\varphi, \varphi \to \psi \vdash \psi$ (modus ponens). These facts are needed for the proof of the completeness theorem. explanation

fol:seq:ppr: prop:provability-land

Proposition seq.1.

fol:seq:ppr: prop:provability-land-left

1. *Both* $\varphi \land \psi \vdash \varphi$ *and* $\varphi \land \psi \vdash \psi$.

fol:seq:ppr: prop:provability-land-right

2. $\varphi, \psi \vdash \varphi \land \psi$.

Proof. 1. Both sequents $\varphi \land \psi \Rightarrow \varphi$ and $\varphi \land \psi \Rightarrow \psi$ are derivable:

$$\frac{\varphi \Rightarrow \varphi}{\varphi \land \psi \Rightarrow \varphi}\land\text{L} \qquad \frac{\psi \Rightarrow \psi}{\varphi \land \psi \Rightarrow \psi}\land\text{L}$$

2. Here is a derivation of the sequent $\varphi, \psi \Rightarrow \varphi \land \psi$:

$$\frac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \varphi \land \psi}\land\text{R}$$

□

fol:seq:ppr: prop:provability-lor

Proposition seq.2.

1. $\varphi \lor \psi, \neg\varphi, \neg\psi$ *is inconsistent.*

2. *Both* $\varphi \vdash \varphi \lor \psi$ *and* $\psi \vdash \varphi \lor \psi$.

Proof. 1. We give a derivation of the sequent $\varphi \lor \psi, \neg\varphi, \neg\psi \Rightarrow$:

$$\frac{\genfrac{}{}{3pt}{}{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\neg\text{L}}{\varphi, \neg\varphi, \neg\psi \Rightarrow} \qquad \genfrac{}{}{3pt}{}{\dfrac{\psi \Rightarrow \psi}{\neg\psi, \psi \Rightarrow}\neg\text{L}}{\psi, \neg\varphi, \neg\psi \Rightarrow}}{\varphi \lor \psi, \neg\varphi, \neg\psi \Rightarrow}\lor\text{L}$$

(Recall that double inference lines indicate several weakening, contraction, and exchange inferences.)

2. Both sequents $\varphi \Rightarrow \varphi \lor \psi$ and $\psi \Rightarrow \varphi \lor \psi$ have derivations:

$$\frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \lor \psi}\lor\text{R} \qquad \frac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi \lor \psi}\lor\text{R}$$

□

fol:seq:ppr: prop:provability-lif

Proposition seq.3.

fol:seq:ppr: prop:provability-lif-left

1. $\varphi, \varphi \to \psi \vdash \psi$.

2. Both $\neg\varphi \vdash \varphi \to \psi$ *and* $\psi \vdash \varphi \to \psi$. *fol:seq:ppr: prop:provability-lif-right*

Proof. 1. The sequent $\varphi \to \psi, \varphi \Rightarrow \psi$ is derivable:

$$\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi \to \psi, \varphi \Rightarrow \psi}\,{\to}\mathrm{L}$$

2. Both sequents $\neg\varphi \Rightarrow \varphi \to \psi$ and $\psi \Rightarrow \varphi \to \psi$ are derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\,\neg\mathrm{L}}{\varphi, \neg\varphi \Rightarrow}\,\mathrm{XL}}{\varphi, \neg\varphi \Rightarrow \psi}\,\mathrm{WR}}{\neg\varphi \Rightarrow \varphi \to \psi}\,{\to}\mathrm{R} \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi}\,\mathrm{WL}}{\psi \Rightarrow \varphi \to \psi}\,{\to}\mathrm{R}$$

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Bibliography