seq.1 Derivability and the Propositional Connectives

fol:seq:ppr:sec

Proposition seq.1.

fol:seq:ppr:prop:provability-land

1. *Both* $\varphi \wedge \psi \vdash \varphi$ *and* $\varphi \wedge \psi \vdash \psi$.

fol:seq:ppr:prop:provability-land-left

fol:seq:ppr:prop:provability-land-right

2. $\varphi, \psi \vdash \varphi \wedge \psi$.

Proof. 1. Both sequents $\varphi \wedge \psi \Rightarrow \varphi$ and $\varphi \wedge \psi \Rightarrow \psi$ are derivable:

$$\dfrac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi}\,\wedge\mathrm{L} \qquad \dfrac{\psi \Rightarrow \psi}{\varphi \wedge \psi \Rightarrow \psi}\,\wedge\mathrm{L}$$

2. Here is a derivation of the sequent $\varphi, \psi \Rightarrow \varphi \wedge \psi$:

$$\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \varphi \wedge \psi}\,\wedge\mathrm{R}$$

□

fol:seq:ppr:prop:provability-lor

Proposition seq.2.

1. $\varphi \vee \psi, \neg\varphi, \neg\psi$ *is inconsistent.*

2. *Both* $\varphi \vdash \varphi \vee \psi$ *and* $\psi \vdash \varphi \vee \psi$.

Proof. 1. We give a derivation of the sequent $\varphi \vee \psi, \neg\varphi, \neg\psi \Rightarrow$:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\,\neg\mathrm{L}}{\varphi, \neg\varphi, \neg\psi \Rightarrow} \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\neg\psi, \psi \Rightarrow}\,\neg\mathrm{L}}{\psi, \neg\varphi, \neg\psi \Rightarrow}}{\varphi \vee \psi, \neg\varphi, \neg\psi \Rightarrow}\,\vee\mathrm{L}$$

(Recall that double inference lines indicate several weakening, contraction, and exchange inferences.)

2. Both sequents $\varphi \Rightarrow \varphi \vee \psi$ and $\psi \Rightarrow \varphi \vee \psi$ have derivations:

$$\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee \psi}\,\vee\mathrm{R} \qquad \dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi \vee \psi}\,\vee\mathrm{R}$$

□

fol:seq:ppr:prop:provability-lif

Proposition seq.3.

fol:seq:ppr:prop:provability-lif-left

1. $\varphi, \varphi \to \psi \vdash \psi$.

fol:seq:ppr:prop:provability-lif-right

2. *Both* $\neg\varphi \vdash \varphi \to \psi$ *and* $\psi \vdash \varphi \to \psi$.

Proof. 1. The sequent $\varphi \to \psi, \varphi \Rightarrow \psi$ is derivable:

$$\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi \to \psi, \varphi \Rightarrow \psi} \to\text{L}$$

2. Both sequents $\neg\varphi \Rightarrow \varphi \to \psi$ and $\psi \Rightarrow \varphi \to \psi$ are derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\,\neg\text{L}}{\varphi, \neg\varphi \Rightarrow}\,\text{XL}}{\varphi, \neg\varphi \Rightarrow \psi}\,\text{WR}}{\neg\varphi \Rightarrow \varphi \to \psi} \to\text{R} \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi}\,\text{WL}}{\psi \Rightarrow \varphi \to \psi} \to\text{R}$$

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Bibliography