We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

**Proposition seq.1.** If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.

*Proof.* There are finite $\Gamma_0$ and $\Gamma_1 \subseteq \Gamma$ such that LK derives $\Gamma_0 \vdash \varphi$ and $\varphi, \Gamma_1 \Rightarrow$. Let the LK-derivation of $\Gamma_0 \vdash \varphi$ be $\pi_0$ and the LK-derivation of $\Gamma_1, \varphi \Rightarrow$ be $\pi_1$. We can then derive

$$\pi_0 \vdash \varphi \quad \pi_1 \vdash \varphi, \Gamma_1 \Rightarrow$$

Using the cut rule, we get

$$\Gamma_0, \Gamma_1 \Rightarrow$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, this shows that $\Gamma$ is inconsistent. \qed

**Proposition seq.2.** $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg \varphi\}$ is inconsistent.

*Proof.* First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation $\pi_0$ of $\Gamma \Rightarrow \varphi$. By adding a $\neg$-L rule, we obtain a derivation of $\neg \varphi, \Gamma \Rightarrow$, i.e., $\Gamma \cup \{\neg \varphi\}$ is inconsistent.

If $\Gamma \cup \{\neg \varphi\}$ is inconsistent, there is a derivation $\pi_1$ of $\neg \varphi, \Gamma \Rightarrow$. The following is a derivation of $\Gamma \Rightarrow \varphi$:

$$\varphi \Rightarrow \varphi \quad \neg \varphi, \pi_0 \Rightarrow \varphi, \Gamma_1 \Rightarrow$$

Using the cut rule, we get

$$\Gamma \Rightarrow \varphi$$

\qed

**Problem seq.1.** Prove that $\Gamma \vdash \neg \varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

**Proposition seq.3.** If $\Gamma \vdash \varphi$ and $\neg \varphi \in \Gamma$, then $\Gamma$ is inconsistent.

*Proof.* Suppose $\Gamma \vdash \varphi$ and $\neg \varphi \in \Gamma$. Then there is a derivation $\pi$ of a sequent $\Gamma_0 \Rightarrow \varphi$. The sequent $\neg \varphi, \Gamma_0 \Rightarrow$ is also derivable:

$$\pi \vdash \varphi \quad \neg \varphi, \varphi \Rightarrow \neg \varphi \Rightarrow \neg \varphi, \Gamma \Rightarrow$$

Using the cut rule, we get

$$\Gamma \Rightarrow \varphi$$

Since $\neg \varphi \in \Gamma$ and $\neg \varphi \in \Gamma$, this shows that $\Gamma$ is inconsistent. \qed
**Proposition seq.4.** If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg \varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.

*Proof.* There are finite sets $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$ and LK-derivations $\pi_0$ and $\pi_1$ of $\varphi, \Gamma_0 \Rightarrow$ and $\neg \varphi, \Gamma_1 \Rightarrow$, respectively. We can then derive

\[
\begin{array}{c}
\vdots \\
\varphi, \Gamma_0 \Rightarrow \\
\hline \\
I_0 \Rightarrow \neg \varphi \\
\hline \\
\Gamma_0, \Gamma_1 \Rightarrow \\
\end{array}
\]

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$. Hence $\Gamma$ is inconsistent.  

\[\Box\]

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**Bibliography**