We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

**Proposition seq.1.** If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.

**Proof.** There are finite $\Gamma_0$ and $\Gamma_1 \subseteq \Gamma$ such that LK derives $\Gamma_0 \Rightarrow \varphi$ and $\varphi, \Gamma_1 \Rightarrow$. Let the LK-derivation of $\Gamma_0 \Rightarrow \varphi$ be $\pi_0$ and the LK-derivation of $\Gamma_1, \varphi \Rightarrow$ be $\pi_1$. We can then derive
\[
\begin{array}{c}
\vdots \\
\pi_0 \\
\vdots \\
\Gamma_0 \Rightarrow \varphi \\
\vdots \\
\pi_1 \\
\vdots \\
\varphi, \Gamma_1 \Rightarrow
\end{array}
\]
\[\text{Cut}\]

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence $\Gamma$ is inconsistent. \(\square\)

**Proposition seq.2.** $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg \varphi\}$ is inconsistent.

**Proof.** First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation $\pi_0$ of $\Gamma \Rightarrow \varphi$. By adding a $\neg$ rule, we obtain a derivation of $\neg \varphi, \Gamma \Rightarrow$, i.e., $\Gamma \cup \{\neg \varphi\}$ is inconsistent.

If $\Gamma \cup \{\neg \varphi\}$ is inconsistent, there is a derivation $\pi_1$ of $\neg \varphi, \Gamma \Rightarrow$. The following is a derivation of $\Gamma \Rightarrow \varphi$:
\[
\begin{array}{c}
\varphi \Rightarrow \varphi \\
\Rightarrow \varphi, \neg \varphi \\
\neg \varphi, \Gamma \Rightarrow \\
\vdots \\
\Gamma \Rightarrow \varphi
\end{array}
\]
\[\text{Cut}\]

\(\square\)

**Problem seq.1.** Prove that $\Gamma \vdash \neg \varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

**Proposition seq.3.** If $\Gamma \vdash \varphi$ and $\neg \varphi \in \Gamma$, then $\Gamma$ is inconsistent.

**Proof.** Suppose $\Gamma \vdash \varphi$ and $\neg \varphi \in \Gamma$. Then there is a derivation $\pi$ of a sequent $\Gamma_0 \Rightarrow \varphi$. The sequent $\neg \varphi, \Gamma_0 \Rightarrow$ is also derivable:
\[
\begin{array}{c}
\vdots \\
\pi \\
\vdots \\
\Gamma_0 \Rightarrow \varphi \\
\vdots \\
\neg \varphi, \varphi \Rightarrow \\
\neg \varphi, \neg \varphi \Rightarrow \\
\vdots \\
\Gamma, \neg \varphi \Rightarrow
\end{array}
\]
\[\text{Cut}\]

Since $\neg \varphi \in \Gamma$ and $\Gamma_0 \subseteq \Gamma$, this shows that $\Gamma$ is inconsistent. \(\square\)
Proposition seq.4. If \( \Gamma \cup \{ \varphi \} \) and \( \Gamma \cup \{ \neg \varphi \} \) are both inconsistent, then \( \Gamma \) is inconsistent.

Proof. There are finite sets \( \Gamma_0 \subseteq \Gamma \) and \( \Gamma_1 \subseteq \Gamma \) and LK-derivations \( \pi_0 \) and \( \pi_1 \) of \( \varphi, \Gamma_0 \Rightarrow \) and \( \neg \varphi, \Gamma_1 \Rightarrow \), respectively. We can then derive

\[
\begin{array}{c}
\vdots \\
\varphi, \Gamma_0 \Rightarrow \\
\vdots \\
\vdash \neg \varphi \text{ by R} \\
\vdash \neg \varphi, \Gamma_1 \Rightarrow \\
\vdash \Gamma_0, \Gamma_1 \Rightarrow \text{ Cut}
\end{array}
\]

Since \( \Gamma_0 \subseteq \Gamma \) and \( \Gamma_1 \subseteq \Gamma \), \( \Gamma_0 \cup \Gamma_1 \subseteq \Gamma \). Hence \( \Gamma \) is inconsistent. \( \square \)

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Bibliography