seq.1 Derivability and Consistency

fol:seq:prv: sec

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:seq:prv: prop:provability-contr

Proposition seq.1. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Proof. There are finite Γ_0 and $\Gamma_1 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow \varphi$ and $\varphi, \Gamma_1 \Rightarrow$. Let the **LK**-derivation of $\Gamma_0 \Rightarrow \varphi$ be π_0 and the **LK**-derivation of $\Gamma_1, \varphi \Rightarrow$ be π_1. We can then derive

$$\frac{\begin{matrix}\vdots\,\pi_0\\ \Gamma_0 \Rightarrow \varphi\end{matrix}\qquad\qquad \begin{matrix}\vdots\,\pi_1\\ \varphi, \Gamma_1 \Rightarrow\end{matrix}}{\Gamma_0, \Gamma_1 \Rightarrow}\ \text{Cut}$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence Γ is inconsistent. □

fol:seq:prv: prop:prov-incons

Proposition seq.2. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation π_0 of $\Gamma \Rightarrow \varphi$. By adding a $\neg$L rule, we obtain a derivation of $\neg\varphi, \Gamma \Rightarrow$, i.e., $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

If $\Gamma \cup \{\neg A\}$ is inconsistent, there is a derivation π_1 of $\neg\varphi, \Gamma \Rightarrow$. The following is a derivation of $\Gamma \Rightarrow \varphi$:

$$\frac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\ \neg\text{R}\qquad\qquad \begin{matrix}\vdots\,\pi_1\\ \neg\varphi, \Gamma \Rightarrow\end{matrix}}{\Gamma \Rightarrow \varphi}\ \text{Cut}$$

□

Problem seq.1. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:seq:prv: prop:explicit-inc

Proposition seq.3. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a derivation π of a sequent $\Gamma_0 \Rightarrow \varphi$. The sequent $\neg\varphi, \Gamma_0 \Rightarrow$ is also derivable:

$$\frac{\begin{matrix}\vdots\,\pi\\ \Gamma_0 \Rightarrow \varphi\end{matrix}\qquad\qquad \dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\ \neg\text{L}}{\varphi, \neg\varphi \Rightarrow}\ \text{XL}}{\Gamma, \neg\varphi \Rightarrow}\ \text{Cut}$$

Since $\neg\varphi \in \Gamma$ and $\Gamma_0 \subseteq \Gamma$, this shows that Γ is inconsistent. □

Proposition seq.4. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.* fol:seq:prv: prop:provability-exhaustive

Proof. There are finite sets $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$ and **LK**-derivations π_0 and π_1 of $\varphi, \Gamma_0 \Rightarrow$ and $\neg\varphi, \Gamma_1 \Rightarrow$, respectively. We can then derive

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_0\\ \varphi, \Gamma_0 \Rightarrow\end{array}}{\Gamma_0 \Rightarrow \neg\varphi}\,\neg\mathrm{R} \qquad \begin{array}{c}\vdots\,\pi_1\\ \neg\varphi, \Gamma_1 \Rightarrow\end{array}}{\Gamma_0, \Gamma_1 \Rightarrow}\,\mathrm{Cut}$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$. Hence Γ is inconsistent. □

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Bibliography