

## seq.1 Derivability and Consistency

fol:seq:prv: sec We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:seq:prv: prop:provability-contr **Proposition seq.1.** *If  $\Gamma \vdash \varphi$  and  $\Gamma \cup \{\varphi\}$  is inconsistent, then  $\Gamma$  is inconsistent.*

*Proof.* There are finite  $\Gamma_0$  and  $\Gamma_1 \subseteq \Gamma$  such that **LK** derives  $\Gamma_0 \Rightarrow \varphi$  and  $\varphi, \Gamma_1 \Rightarrow$ . Let the **LK-derivation** of  $\Gamma_0 \Rightarrow \varphi$  be  $\pi_0$  and the **LK-derivation** of  $\Gamma_1, \varphi \Rightarrow$  be  $\pi_1$ . We can then **derive**

$$\frac{\begin{array}{c} \vdots \\ \pi_0 \\ \vdots \\ \Gamma_0 \Rightarrow \varphi \end{array} \quad \begin{array}{c} \vdots \\ \pi_1 \\ \vdots \\ \varphi, \Gamma_1 \Rightarrow \end{array}}{\Gamma_0, \Gamma_1 \Rightarrow} \text{Cut}$$

Since  $\Gamma_0 \subseteq \Gamma$  and  $\Gamma_1 \subseteq \Gamma$ ,  $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$ , hence  $\Gamma$  is inconsistent.  $\square$

fol:seq:prv: prop:prov-incons **Proposition seq.2.**  *$\Gamma \vdash \varphi$  iff  $\Gamma \cup \{\neg\varphi\}$  is inconsistent.*

*Proof.* First suppose  $\Gamma \vdash \varphi$ , i.e., there is a **derivation**  $\pi_0$  of  $\Gamma \Rightarrow \varphi$ . By adding a  $\neg$ -L rule, we obtain a **derivation** of  $\neg\varphi, \Gamma \Rightarrow$ , i.e.,  $\Gamma \cup \{\neg\varphi\}$  is inconsistent.

If  $\Gamma \cup \{\neg\varphi\}$  is inconsistent, there is a **derivation**  $\pi_1$  of  $\neg\varphi, \Gamma \Rightarrow$ . The following is a **derivation** of  $\Gamma \Rightarrow \varphi$ :

$$\frac{\frac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi} \neg\text{R} \quad \begin{array}{c} \vdots \\ \pi_1 \\ \vdots \\ \neg\varphi, \Gamma \Rightarrow \end{array}}{\Gamma \Rightarrow \varphi} \text{Cut}$$

$\square$

**Problem seq.1.** Prove that  $\Gamma \vdash \neg\varphi$  iff  $\Gamma \cup \{\varphi\}$  is inconsistent.

fol:seq:prv: prop:explicit-inc **Proposition seq.3.** *If  $\Gamma \vdash \varphi$  and  $\neg\varphi \in \Gamma$ , then  $\Gamma$  is inconsistent.*

*Proof.* Suppose  $\Gamma \vdash \varphi$  and  $\neg\varphi \in \Gamma$ . Then there is a **derivation**  $\pi$  of a sequent  $\Gamma_0 \Rightarrow \varphi$ . The sequent  $\neg\varphi, \Gamma_0 \Rightarrow$  is also **derivable**:

$$\frac{\begin{array}{c} \vdots \\ \pi \\ \vdots \\ \Gamma_0 \Rightarrow \varphi \end{array} \quad \frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg\text{L}}{\varphi, \neg\varphi \Rightarrow} \text{XL}}{\Gamma, \neg\varphi \Rightarrow} \text{Cut}$$

Since  $\neg\varphi \in \Gamma$  and  $\Gamma_0 \subseteq \Gamma$ , this shows that  $\Gamma$  is inconsistent.  $\square$

**Proposition seq.4.** *If  $\Gamma \cup \{\varphi\}$  and  $\Gamma \cup \{\neg\varphi\}$  are both inconsistent, then  $\Gamma$  is inconsistent.* fol:seq:prv:  
prop:provability-exhaustive

*Proof.* There are finite sets  $\Gamma_0 \subseteq \Gamma$  and  $\Gamma_1 \subseteq \Gamma$  and **LK-derivations**  $\pi_0$  and  $\pi_1$  of  $\varphi, \Gamma_0 \Rightarrow$  and  $\neg\varphi, \Gamma_1 \Rightarrow$ , respectively. We can then **derive**

$$\frac{\frac{\begin{array}{c} \vdots \\ \vdots \pi_0 \\ \vdots \\ \varphi, \Gamma_0 \Rightarrow \end{array}}{\Gamma_0 \Rightarrow \neg\varphi} \text{ } \neg\text{R} \quad \frac{\begin{array}{c} \vdots \\ \vdots \pi_1 \\ \vdots \\ \neg\varphi, \Gamma_1 \Rightarrow \end{array}}{\neg\varphi, \Gamma_1 \Rightarrow} \text{ } \text{Cut}}{\Gamma_0, \Gamma_1 \Rightarrow} \text{Cut}$$

Since  $\Gamma_0 \subseteq \Gamma$  and  $\Gamma_1 \subseteq \Gamma$ ,  $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$ . Hence  $\Gamma$  is inconsistent. □

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## Bibliography