

seq.1 Derivability and Consistency

fol:seq:prv:sec We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:seq:prv:prop:provability-contr **Proposition seq.1.** *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Proof. There are finite Γ_0 and $\Gamma_1 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow \varphi$ and $\varphi, \Gamma_1 \Rightarrow$. Let the **LK-derivation** of $\Gamma_0 \Rightarrow \varphi$ be π_0 and the **LK-derivation** of $\Gamma_1, \varphi \Rightarrow$ be π_1 . We can then **derive**

$$\frac{\begin{array}{c} \vdots \\ \vdots \pi_0 \\ \vdots \\ \Gamma_0 \Rightarrow \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \pi_1 \\ \vdots \\ \varphi, \Gamma_1 \Rightarrow \end{array}}{\Gamma_0, \Gamma_1 \Rightarrow} \text{Cut}$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence Γ is inconsistent. □

fol:seq:prv:prop:prov-incons **Proposition seq.2.** *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a **derivation** π_0 of $\Gamma \Rightarrow \varphi$. By adding a \neg L rule, we obtain a **derivation** of $\neg\varphi, \Gamma \Rightarrow$, i.e., $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

If $\Gamma \cup \{\neg\varphi\}$ is inconsistent, there is a **derivation** π_1 of $\neg\varphi, \Gamma \Rightarrow$. The following is a **derivation** of $\Gamma \Rightarrow \varphi$:

$$\frac{\frac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi} \neg R \quad \begin{array}{c} \vdots \\ \vdots \pi_1 \\ \vdots \\ \neg\varphi, \Gamma \Rightarrow \end{array}}{\Gamma \Rightarrow \varphi} \text{Cut} \quad \square$$

Problem seq.1. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:seq:prv:prop:explicit-inc **Proposition seq.3.** *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a **derivation** π of a sequent $\Gamma_0 \Rightarrow \varphi$. The sequent $\neg\varphi, \Gamma_0 \Rightarrow$ is also **derivable**:

$$\frac{\begin{array}{c} \vdots \\ \vdots \pi \\ \vdots \\ \Gamma_0 \Rightarrow \varphi \end{array} \quad \frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg L}{\varphi, \neg\varphi \Rightarrow} \text{XL}}{\Gamma, \neg\varphi \Rightarrow} \text{Cut}$$

Since $\neg\varphi \in \Gamma$ and $\Gamma_0 \subseteq \Gamma$, this shows that Γ is inconsistent. □

Proposition seq.4. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.* fol:seq:prv:
prop:provability-exhaustive

Proof. There are finite sets $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$ and **LK-derivations** π_0 and π_1 of $\varphi, \Gamma_0 \Rightarrow$ and $\neg\varphi, \Gamma_1 \Rightarrow$, respectively. We can then **derive**

$$\frac{\frac{\frac{\vdots \pi_0}{\varphi, \Gamma_0 \Rightarrow}}{\Gamma_0 \Rightarrow \neg\varphi} \neg R \quad \frac{\vdots \pi_1}{\neg\varphi, \Gamma_1 \Rightarrow}}{\Gamma_0, \Gamma_1 \Rightarrow} \text{Cut}$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$. Hence Γ is inconsistent. □

Photo Credits

Bibliography