seq.1 Derivability and Consistency

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

Proposition seq.1. If \( \Gamma \vdash \varphi \) and \( \Gamma \cup \{ \varphi \} \) is inconsistent, then \( \Gamma \) is inconsistent.

Proof. There are finite \( \Gamma_0 \) and \( \Gamma_1 \subseteq \Gamma \) such that LK derives \( \Gamma_0 \Rightarrow \varphi \) and \( \varphi, \Gamma_1 \Rightarrow \). Let the LK-derivation of \( \Gamma_0 \Rightarrow \varphi \) be \( \pi_0 \) and the LK-derivation of \( \varphi, \Gamma_1 \Rightarrow \) be \( \pi_1 \). We can then derive

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\Gamma_0 \Rightarrow \varphi \\
\Gamma_0, \Gamma_1 \Rightarrow \text{Cut}
\end{array}
\]

Since \( \Gamma_0 \subseteq \Gamma \) and \( \Gamma_1 \subseteq \Gamma \), \( \Gamma_0 \cup \Gamma_1 \subseteq \Gamma \), hence \( \Gamma \) is inconsistent. \( \square \)

Proposition seq.2. \( \Gamma \vdash \varphi \) iff \( \Gamma \cup \{ \neg \varphi \} \) is inconsistent.

Proof. First suppose \( \Gamma \vdash \varphi \), i.e., there is a derivation \( \pi_0 \) of \( \Gamma \Rightarrow \varphi \). By adding a \( \neg\)L rule, we obtain a derivation of \( \neg \varphi, \Gamma \Rightarrow \), i.e., \( \Gamma \cup \{ \neg \varphi \} \) is inconsistent.

If \( \Gamma \cup \{ \neg A \} \) is inconsistent, there is a derivation \( \pi_1 \) of \( \neg \varphi, \Gamma \Rightarrow \). The following is a derivation of \( \Gamma \Rightarrow \varphi \):

\[
\begin{array}{c}
\varphi \Rightarrow \varphi \\
\neg \varphi \Rightarrow \varphi, \neg \varphi \\
\Gamma \Rightarrow \varphi \text{ Cut}
\end{array}
\]

\( \square \)

Problem seq.1. Prove that \( \Gamma \vdash \neg \varphi \) iff \( \Gamma \cup \{ \varphi \} \) is inconsistent.

Proposition seq.3. If \( \Gamma \vdash \varphi \) and \( \neg \varphi \in \Gamma \), then \( \Gamma \) is inconsistent.

Proof. Suppose \( \Gamma \vdash \varphi \) and \( \neg \varphi \in \Gamma \). Then there is a derivation \( \pi \) of a sequent \( \Gamma_0 \Rightarrow \varphi \). The sequent \( \neg \varphi, \Gamma_0 \Rightarrow \) is also derivable:

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\varphi \Rightarrow \varphi \\
\neg \varphi, \Gamma \Rightarrow \text{Cut}
\end{array}
\]

Since \( \neg \varphi \in \Gamma \) and \( \Gamma_0 \subseteq \Gamma \), this shows that \( \Gamma \) is inconsistent. \( \square \)
Proposition seq.4. If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg \varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.

Proof. There are finite sets $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$ and LK-derivations $\pi_0$ and $\pi_1$ of $\varphi, \Gamma_0 \Rightarrow$ and $\neg \varphi, \Gamma_1 \Rightarrow$, respectively. We can then derive

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\varphi, \Gamma_0 \Rightarrow \\
\Gamma_0 \Rightarrow \neg \varphi \\
\neg \varphi, \Gamma_1 \Rightarrow \\
\Gamma_0, \Gamma_1 \Rightarrow \\
\end{array}
\]

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$. Hence $\Gamma$ is inconsistent. \qed

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Bibliography