

This section collects the definitions of the provability relation and consistency for natural deduction.

## seq.1 Proof-Theoretic Notions

fol:seq:ptn: Just as we've defined a number of important semantic notions (validity, explanation, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the **derivability** or **non-derivability** of certain sequents. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorem*.

**Definition seq.1** (Theorems). A **sentence**  $\varphi$  is a *theorem* if there is a **derivation** in **LK** of the sequent  $\Rightarrow \varphi$ . We write  $\vdash \varphi$  if  $\varphi$  is a theorem and  $\not\vdash \varphi$  if it is not.

**Definition seq.2** (Derivability). A **sentence**  $\varphi$  is *derivable* from a set of **sentences**  $\Gamma$ ,  $\Gamma \vdash \varphi$ , iff there is a finite subset  $\Gamma_0 \subseteq \Gamma$  and a sequence  $\Gamma'_0$  of the **sentences** in  $\Gamma_0$  such that **LK** derives  $\Gamma'_0 \Rightarrow \varphi$ . If  $\varphi$  is not **derivable** from  $\Gamma$  we write  $\Gamma \not\vdash \varphi$ .

Because of the contraction, weakening, and exchange rules, the order and number of **sentences** in  $\Gamma'_0$  does not matter: if a sequent  $\Gamma'_0 \Rightarrow \varphi$  is **derivable**, then so is  $\Gamma''_0 \Rightarrow \varphi$  for any  $\Gamma''_0$  that contains the same **sentences** as  $\Gamma'_0$ . For instance, if  $\Gamma_0 = \{\psi, \chi\}$  then both  $\Gamma'_0 = \langle \psi, \psi, \chi \rangle$  and  $\Gamma''_0 = \langle \chi, \chi, \psi \rangle$  are sequences containing just the **sentences** in  $\Gamma_0$ . If a sequent containing one is **derivable**, so is the other, e.g.:

$$\begin{array}{c}
 \vdots \\
 \vdots \\
 \vdots \\
 \frac{\psi, \psi, \chi \Rightarrow \varphi}{\psi, \chi \Rightarrow \varphi} \text{CL} \\
 \frac{\psi, \chi \Rightarrow \varphi}{\chi, \psi \Rightarrow \varphi} \text{XL} \\
 \frac{\chi, \psi \Rightarrow \varphi}{\chi, \chi, \psi \Rightarrow \varphi} \text{WL}
 \end{array}$$

From now on we'll say that if  $\Gamma_0$  is a finite set of **sentences** then  $\Gamma_0 \Rightarrow \varphi$  is any sequent where the antecedent is a sequence of **sentences** in  $\Gamma_0$  and tacitly include contractions, exchanges, and weakenings if necessary.

**Definition seq.3** (Consistency). A set of sentences  $\Gamma$  is *inconsistent* iff there is a finite subset  $\Gamma_0 \subseteq \Gamma$  such that **LK** derives  $\Gamma_0 \Rightarrow$  . If  $\Gamma$  is not inconsistent, i.e., if for every finite  $\Gamma_0 \subseteq \Gamma$ , **LK** does not **derive**  $\Gamma_0 \Rightarrow$  , we say it is *consistent*.

fol:seq:ptn: **Proposition seq.4** (Reflexivity). If  $\varphi \in \Gamma$ , then  $\Gamma \vdash \varphi$ .  
prop:reflexivity

