seq.1 Proof-Theoretic Notions

fol:seq:ptn:sec

Just as we've defined a number of important semantic notions (validity, entailment, satisfiabilty), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain sequents. It was an important discovery, due to Gödel, that these notions coincide. That they do is the content of the *completeness theorem*. explanation

Definition seq.1 (Theorems)**.** A sentence φ is a *theorem* if there is a derivation in **LK** of the sequent $\Rightarrow \varphi$. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Definition seq.2 (Derivability)**.** A sentence φ is *derivable from* a set of sentences Γ, $\Gamma \vdash \varphi$, iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ and a sequence Γ_0' of the sentences in Γ_0 such that **LK** derives $\Gamma_0' \Rightarrow \varphi$. If φ is not derivable from Γ we write $\Gamma \nvdash \varphi$.

Because of the contraction, weakening, and exchange rules, the order and number of sentences in Γ_0' does not matter: if a sequent $\Gamma_0' \Rightarrow \varphi$ is derivable, then so is $\Gamma_0'' \Rightarrow \varphi$ for any Γ_0'' that contains the same sentences as Γ_0'. For instance, if $\Gamma_0 = \{\psi, \chi\}$ then both $\Gamma_0' = \langle \psi, \psi, \chi \rangle$ and $\Gamma_0'' = \langle \chi, \chi, \psi \rangle$ are sequences containing just the sentences in Γ_0. If a sequent containing one is derivable, so is the other, e.g.:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \psi, \psi, \chi \Rightarrow \varphi\end{array}}{\psi, \chi \Rightarrow \varphi}\ \text{CL}}{\chi, \psi \Rightarrow \varphi}\ \text{XL}}{\chi, \chi, \psi \Rightarrow \varphi}\ \text{WL}$$

From now on we'll say that if Γ_0 is a finite set of sentences then $\Gamma_0 \Rightarrow \varphi$ is any sequent where the antecedent is a sequence of sentences in Γ_0 and tacitly include contractions, exchanges, and weakenings if necessary.

Definition seq.3 (Consistency)**.** A set of sentences Γ is *inconsistent* iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow$. If Γ is not inconsistent, i.e., if for every finite $\Gamma_0 \subseteq \Gamma$, **LK** does not derive $\Gamma_0 \Rightarrow$, we say it is *consistent*.

fol:seq:ptn:prop:reflexivity

Proposition seq.4 (Reflexivity)**.** *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. The initial sequent $\varphi \Rightarrow \varphi$ is derivable, and $\{\varphi\} \subseteq \Gamma$. □

fol:seq:ptn:prop:monotony

Proposition seq.5 (Monotony)**.** *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any finite $\Gamma_0 \subseteq \Gamma$ is also a finite subset of Δ, so a derivation of $\Gamma_0 \Rightarrow \varphi$ also shows $\Delta \vdash \varphi$. □

Proposition seq.6 (Transitivity). *If $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$ and $\Delta \vdash \psi$, then $\Gamma \vdash \psi$.* fol:seq:ptn:prop:transitivity

Proof. If $\Delta \vdash \psi$, then for some finite subset $\Delta_0 \subseteq \Delta$, there is a derivation of $\Delta_0 \Rightarrow \psi$. We show that $\Gamma \vdash \psi$ by induction on the number n of sentences in Δ_0.

If $n = 0$, then Δ_0 is empty, and $\Rightarrow \psi$ is provable. Since $\emptyset \subseteq \Gamma$, $\Gamma \vdash \psi$.

Otherwise, pick $\varphi \in \Delta_0$ and let $\Delta_1 = \Delta_0 \setminus \{\varphi\}$. There is a derivation π_0 of $\varphi, \Delta_1 \Rightarrow \psi$. We obtain the derivation π_1:

$$\dfrac{\begin{array}{c}\vdots\,\pi\\ \varphi, \Delta_1 \Rightarrow \psi\end{array}}{\Delta_1 \Rightarrow \varphi \to \psi}\;{\to}\mathrm{R}$$

Since the number of sentences in Δ_1 is $n-1$, the inductive hypothesis applies: there is a derivation π_2 of $\Gamma_0 \Rightarrow \varphi \to \psi$ for some $\Gamma_0 \subseteq \Gamma$. Since $\Gamma \vdash \varphi$ there is also a derivation π_3 of $\Gamma_1 \Rightarrow \varphi$. Now consider:

$$\dfrac{\begin{array}{c}\vdots\,\pi_2\\ \Gamma_0 \Rightarrow \varphi \to \psi\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_3\\ \Gamma_1 \Rightarrow \varphi\end{array} \qquad \psi \Rightarrow \psi}{\varphi \to \psi, \Gamma_1 \Rightarrow \psi}\;{\to}\mathrm{L}}{\Gamma_0, \Gamma_1 \Rightarrow \psi}\;\mathrm{Cut}$$

This shows $\Gamma \vdash \psi$. □

Proposition seq.7. *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence φ.* fol:seq:ptn:prop:incons

Proof. Exercise. □

Problem seq.1. Prove Proposition seq.7

Proposition seq.8 (Compactness). fol:seq:ptn:prop:proves-compact

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*
2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Rightarrow \varphi$ has a derivation. Consequently, $\Gamma_0 \vdash \varphi$.

2. If Γ is inconsistent, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow$. But then Γ_0 is a finite subset of Γ that is inconsistent.

□

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Bibliography