

seq.1 Proof-Theoretic Notions

fol:seq:ptn: Just as we've defined a number of important semantic notions (validity, explanation
sec entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the **derivability** or **non-derivability** of certain sequents. It was an important discovery, due to Gödel, that these notions coincide. That they do is the content of the *completeness theorem*.

Definition seq.1 (Theorems). A **sentence** φ is a *theorem* if there is a **derivation** in **LK** of the sequent $\Rightarrow \varphi$. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition seq.2 (Derivability). A **sentence** φ is *derivable* from a set of **sentences** Γ , $\Gamma \vdash \varphi$, iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ and a sequence Γ'_0 of the **sentences** in Γ_0 such that **LK** derives $\Gamma'_0 \Rightarrow \varphi$. If φ is not **derivable** from Γ we write $\Gamma \not\vdash \varphi$.

Because of the contraction, weakening, and exchange rules, the order and number of **sentences** in Γ'_0 does not matter: if a sequent $\Gamma'_0 \Rightarrow \varphi$ is **derivable**, then so is $\Gamma''_0 \Rightarrow \varphi$ for any Γ''_0 that contains the same **sentences** as Γ'_0 . For instance, if $\Gamma_0 = \{\psi, \chi\}$ then both $\Gamma'_0 = \langle \psi, \psi, \chi \rangle$ and $\Gamma''_0 = \langle \chi, \chi, \psi \rangle$ are sequences containing just the **sentences** in Γ_0 . If a sequent containing one is **derivable**, so is the other, e.g.:

$$\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \frac{\psi, \psi, \chi \Rightarrow \varphi}{\psi, \chi \Rightarrow \varphi} \text{CL} \\ \frac{\psi, \chi \Rightarrow \varphi}{\chi, \psi \Rightarrow \varphi} \text{XL} \\ \frac{\chi, \psi \Rightarrow \varphi}{\chi, \chi, \psi \Rightarrow \varphi} \text{WL} \end{array}$$

From now on we'll say that if Γ_0 is a finite set of **sentences** then $\Gamma_0 \Rightarrow \varphi$ is any sequent where the antecedent is a sequence of **sentences** in Γ_0 and tacitly include contractions, exchanges, and weakenings if necessary.

Definition seq.3 (Consistency). A set of sentences Γ is *inconsistent* iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow$. If Γ is not inconsistent, i.e., if for every finite $\Gamma_0 \subseteq \Gamma$, **LK** does not **derive** $\Gamma_0 \Rightarrow$, we say it is *consistent*.

fol:seq:ptn: **Proposition seq.4** (Reflexivity). If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.
prop:reflexivity

Proof. The initial sequent $\varphi \Rightarrow \varphi$ is **derivable**, and $\{\varphi\} \subseteq \Gamma$. □

fol:seq:ptn: **Proposition seq.5** (Monotony). If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.
prop:monotony

Proof. Any finite $\Gamma_0 \subseteq \Gamma$ is also a finite subset of Δ , so a **derivation** of $\Gamma_0 \Rightarrow \varphi$ also shows $\Delta \vdash \varphi$. □

Proposition seq.6 (Transitivity). *If $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$ and $\Delta \vdash \psi$, then $\Gamma \vdash \psi$.* *fol:seq:ptn:
prop:transitivity*

Proof. If $\Delta \vdash \psi$, then for some finite subset $\Delta_0 \subseteq \Delta$, there is a derivation of $\Delta_0 \Rightarrow \psi$. We show that $\Gamma \vdash \psi$ by induction on the number n of sentences in Δ_0 .

If $n = 0$, then Δ_0 is empty, and $\Rightarrow \psi$ is provable. Since $\emptyset \subseteq \Gamma$, $\Gamma \vdash \psi$.

Otherwise, pick $\varphi \in \Delta_0$ and let $\Delta_1 = \Delta_0 \setminus \{\varphi\}$. There is a derivation π_0 of $\varphi, \Delta_1 \Rightarrow \psi$. We obtain the derivation π_1 :

$$\frac{\begin{array}{c} \vdots \\ \pi \\ \vdots \end{array} \quad \varphi, \Delta_1 \Rightarrow \psi}{\Delta_1 \Rightarrow \varphi \rightarrow \psi} \rightarrow R$$

Since the number of sentences in Δ_1 is $n - 1$, the inductive hypothesis applies: there is a derivation π_2 of $\Gamma_0 \Rightarrow \varphi \rightarrow \psi$ for some $\Gamma_0 \subseteq \Gamma$. Since $\Gamma \vdash \varphi$ there is also a derivation π_3 of $\Gamma_1 \Rightarrow \varphi$. Now consider:

$$\frac{\begin{array}{c} \vdots \\ \pi_2 \\ \vdots \end{array} \quad \Gamma_0 \Rightarrow \varphi \rightarrow \psi \quad \frac{\begin{array}{c} \vdots \\ \pi_3 \\ \vdots \end{array} \quad \Gamma_1 \Rightarrow \varphi \quad \psi \Rightarrow \psi}{\varphi \rightarrow \psi, \Gamma_1 \Rightarrow \psi} \rightarrow L}{\Gamma_0, \Gamma_1 \Rightarrow \psi} \text{Cut}$$

This shows $\Gamma \vdash \psi$. □

Proposition seq.7. *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence φ .* *fol:seq:ptn:
prop:incons*

Proof. Exercise. □

Problem seq.1. Prove Proposition seq.7

Proposition seq.8 (Compactness). *fol:seq:ptn:
prop:proves-compact*

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of Γ is consistent, then Γ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Rightarrow \varphi$ has a derivation. Consequently, $\Gamma_0 \vdash \varphi$.

2. If Γ is inconsistent, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow$. But then Γ_0 is a finite subset of Γ that is inconsistent. □

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Bibliography