This section collects the definitions of the provability relation and consistency for natural deduction.

seq.1 Proof-Theoretic Notions

Just as we’ve defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding proof-theoretic notions. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain sequents. It was an important discovery that these notions coincide. That they do is the content of the soundness and completeness theorem.

**Definition seq.1 (Theorems).** A sentence $\varphi$ is a *theorem* if there is a derivation in $\text{LK}$ of the sequent $\Gamma \Rightarrow \varphi$. We write $\Gamma \vdash \varphi$ if $\varphi$ is a theorem and $\not \Gamma \varphi$ if it is not.

**Definition seq.2 (Derivability).** A sentence $\varphi$ is *derivable* from a set of sentences $\Gamma$, $\Gamma \vdash \varphi$, iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ and a sequence $\Gamma'_0$ of the sentences in $\Gamma_0$ such that $\text{LK}$ derives $\Gamma'_0 \Rightarrow \varphi$. If $\varphi$ is not derivable from $\Gamma$ we write $\Gamma \not \Rightarrow \varphi$.

Because of the contraction, weakening, and exchange rules, the order and number of sentences in $\Gamma'_0$ does not matter: if a sequent $\Gamma'_0 \Rightarrow \varphi$ is derivable, then so is $\Gamma''_0 \Rightarrow \varphi$ for any $\Gamma''_0$ that contains the same sentences as $\Gamma'_0$. For instance, if $\Gamma_0 = \{\psi, \chi\}$ then both $\Gamma'_0 = \langle \psi, \psi, \chi \rangle$ and $\Gamma''_0 = \langle \chi, \chi, \psi \rangle$ are sequences containing just the sentences in $\Gamma_0$. If a sequent containing one is derivable, so is the other, e.g.:

\[ \begin{array}{c}
  \ldots \\
  \psi, \psi, \chi \Rightarrow \varphi \quad \text{CL} \\
  \psi, \chi \Rightarrow \varphi \quad \text{XL} \\
  \chi, \psi \Rightarrow \varphi \quad \text{WL} \\
  \chi, \chi, \psi \Rightarrow \varphi
\end{array} \]

From now on we’ll say that if $\Gamma_0$ is a finite set of sentences then $\Gamma_0 \Rightarrow \varphi$ is any sequent where the antecedent is a sequence of sentences in $\Gamma_0$ and tacitly include contractions, exchanges, and weakenings if necessary.

**Definition seq.3 (Consistency).** A set of sentences $\Gamma$ is *inconsistent* iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\text{LK}$ derives $\Gamma_0 \Rightarrow \varphi$. If $\Gamma$ is not inconsistent, i.e., if for every finite $\Gamma_0 \subseteq \Gamma$, $\text{LK}$ does not derive $\Gamma_0 \Rightarrow \varphi$, we say it is *consistent*.

**Proposition seq.4 (Reflexivity).** If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$. 

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Proof. The initial sequent $\varphi \Rightarrow \varphi$ is derivable, and $\{\varphi\} \subseteq \Gamma$.

Proposition seq.5 (Monotony). If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.

Proof. Suppose $\Gamma \vdash \varphi$, i.e., there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \Rightarrow \varphi$ is derivable. Since $\Gamma \subseteq \Delta$, then $\Gamma_0$ is also a finite subset of $\Delta$. The derivation of $\Gamma_0 \Rightarrow \varphi$ thus also shows $\Delta \vdash \varphi$.

Proposition seq.6 (Transitivity). If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.

Proof. If $\Gamma \vdash \varphi$, there is a finite $\Gamma_0 \subseteq \Gamma$ and a derivation $\pi_0$ of $\Gamma_0 \Rightarrow \varphi$. If $\{\varphi\} \cup \Delta \vdash \psi$, then for some finite subset $\Delta_0 \subseteq \Delta$, there is a derivation $\pi_1$ of $\varphi, \Delta_0 \Rightarrow \psi$. Consider the following derivation:

$$\begin{array}{c}
\vdots \pi_0 \vdots \pi_1 \\
\Gamma_0 \Rightarrow \varphi, \varphi, \Delta_0 \Rightarrow \psi \\
\Gamma_0, \Delta_0 \Rightarrow \psi
\end{array}$$

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Since $\Gamma_0 \cup \Delta_0 \subseteq \Gamma \cup \Delta$, this shows $\Gamma \cup \Delta \vdash \psi$.

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \ldots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each $i$, then $\Gamma \vdash \psi$.

Proposition seq.7. $\Gamma$ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence $\varphi$.

Proof. Exercise.

Problem seq.1. Prove Proposition seq.7

Proposition seq.8 (Compactness).

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.

2. If every finite subset of $\Gamma$ is consistent, then $\Gamma$ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Rightarrow \varphi$ has a derivation. Consequently, $\Gamma_0 \vdash \varphi$.

2. If $\Gamma$ is inconsistent, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that LK derives $\Gamma_0 \Rightarrow \varphi$. But then $\Gamma_0$ is a finite subset of $\Gamma$ that is inconsistent.

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Bibliography