Derivations with Identity predicate

Derivations with identity predicate require additional initial sequents and inference rules.

Definition seq.1 (Initial sequents for =). If $t$ is a closed term, then $\Rightarrow t = t$ is an initial sequent.

The rules for $=$ are ($t_1$ and $t_2$ are closed terms):

\[
\begin{align*}
  t_1 = t_2, \Gamma & \Rightarrow \Delta, \varphi(t_1) = t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2) = \frac{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)}{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)}.
\end{align*}
\]

Example seq.2. If $s$ and $t$ are closed terms, then $s = t, \varphi(s) \vdash \varphi(t)$:

\[
\begin{align*}
  \varphi(s) \Rightarrow \varphi(s) \quad WL \quad & \quad s = t, \varphi(s) \Rightarrow \varphi(s) \quad WL \quad & \quad s = t, \varphi(s) \Rightarrow \varphi(t) \quad WL.
\end{align*}
\]

This may be familiar as the principle of substitutability of identicals, or Leibniz’ Law.

$LK$ proves that $=$ is symmetric and transitive:

\[
\begin{align*}
  & \Rightarrow t_1 = t_1 \quad WL \quad & \quad t_1 = t_2 \Rightarrow t_1 = t_2 \quad WL \quad & \quad t_2 = t_3, t_1 = t_2 \Rightarrow t_1 = t_3 \quad XL.
  & t_1 = t_2 \Rightarrow t_2 = t_1 \quad WL \quad & \quad t_2 = t_3, t_1 = t_2 \Rightarrow t_1 = t_3 \quad XL \quad & \quad t_1 = t_2, t_2 = t_3 \Rightarrow t_1 = t_3 \quad XL.
\end{align*}
\]

In the proof on the left, the formula $x = t_1$ is our $\varphi(x)$. On the right, we take $\varphi(x)$ to be $t_1 = x$.

Problem seq.1. Give derivations of the following sequents:

1. $\Rightarrow \forall x \forall y ((x = y \land \varphi(x)) \rightarrow \varphi(y))$

2. $\exists x \varphi(x) \land \forall y \forall z ((\varphi(y) \land \varphi(z)) \rightarrow y = z) \Rightarrow \exists x (\varphi(x) \land \forall y (\varphi(y) \rightarrow y = x))$

Photo Credits

Bibliography