seq.1 Derivations with Identity predicate

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Derivations with identity predicate require additional initial sequents and inference rules.

Definition seq.1 (Initial sequents for $=$)**.** If t is a closed term, then $\Rightarrow t = t$ is an initial sequent.

The rules for $=$ are (t_1 and t_2 are closed terms):

$$\frac{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)}{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)} = \qquad \frac{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)}{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)} =$$

Example seq.2. If s and t are closed terms, then $s = t, \varphi(s) \vdash \varphi(t)$:

$$\cfrac{\cfrac{\varphi(s) \Rightarrow \varphi(s)}{s = t, \varphi(s) \Rightarrow \varphi(s)}\,\mathrm{WL}}{s = t, \varphi(s) \Rightarrow \varphi(t)}\,=$$

This may be familiar as the principle of substitutability of identicals, or Leibniz' Law.

LK proves that $=$ is symmetric and transitive:

$$\cfrac{\cfrac{\Rightarrow t_1 = t_1}{t_1 = t_2 \Rightarrow t_1 = t_1}\,\mathrm{WL}}{t_1 = t_2 \Rightarrow t_2 = t_1}\,= \qquad \cfrac{\cfrac{\cfrac{t_1 = t_2 \Rightarrow t_1 = t_2}{t_2 = t_3, t_1 = t_2 \Rightarrow t_1 = t_2}\,\mathrm{WL}}{t_2 = t_3, t_1 = t_2 \Rightarrow t_1 = t_3}\,=}{t_1 = t_2, t_2 = t_3 \Rightarrow t_1 = t_3}\,\mathrm{XL}$$

In the proof on the left, the formula $x = t_1$ is our $\varphi(x)$. On the right, we take $\varphi(x)$ to be $t_1 = x$.

Problem seq.1. Give derivations of the following sequents:

1. $\Rightarrow \forall x\, \forall y\, ((x = y \land \varphi(x)) \rightarrow \varphi(y))$
2. $\exists x\, \varphi(x) \land \forall y\, \forall z\, ((\varphi(y) \land \varphi(z)) \rightarrow y = z) \Rightarrow \exists x\, (\varphi(x) \land \forall y\, (\varphi(y) \rightarrow y = x))$

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Bibliography