Derivations with Identity predicate

Derivations with identity predicate require additional initial sequents and inference rules.

**Definition seq.1** (Initial sequents for \( = \)). If \( t \) is a closed term, then \( \Rightarrow t = t \) is an initial sequent.

The rules for \( = \) are (\( t_1 \) and \( t_2 \) are closed terms):

\[
\begin{align*}
& t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1) = t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2) = \\
& t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1) = t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2) =
\end{align*}
\]

**Example seq.2.** If \( s \) and \( t \) are closed terms, then \( s = t, \varphi(s) \vdash \varphi(t) \):

\[
\begin{align*}
& \varphi(s) \Rightarrow \varphi(s) \quad \text{WL} \\
& s = t, \varphi(s) \Rightarrow \varphi(s) \quad \text{WL} \\
& s = t, \varphi(s) \Rightarrow \varphi(t) =
\end{align*}
\]

This may be familiar as the principle of substitutability of identicals, or Leibniz’ Law.

**LK** proves that \( = \) is symmetric and transitive:

\[
\begin{align*}
& \Rightarrow t_1 = t_1 \quad \text{WL} \\
& t_1 = t_2 \Rightarrow t_1 = t_1 = t_2 \quad \text{WL} \\
& t_2 = t_1, t_1 = t_2 \Rightarrow t_1 = t_2 = t_3 \quad \text{XL}
\end{align*}
\]

In the proof on the left, the formula \( x = t_1 \) is our \( \varphi(x) \). On the right, we take \( \varphi(x) \) to be \( t_1 = x \).

**Problem seq.1.** Give derivations of the following sequents:

1. \( \Rightarrow \forall x \forall y ((x = y \land \varphi(x)) \rightarrow \varphi(y)) \)

2. \( \exists x \varphi(x) \land \forall y \forall z ((\varphi(y) \land \varphi(z)) \rightarrow y = z) \Rightarrow \exists x (\varphi(x) \land \forall y (\varphi(y) \rightarrow y = x)) \)

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**Bibliography**