Derivations with Identity Predicate

Derivations with identity predicate require additional initial sequents and inference rules.

Definition seq.1 (Initial sequents for $=$). If $t$ is a closed term, then $\Rightarrow t = t$ is an initial sequent.

The rules for $=$ are ($t_1$ and $t_2$ are closed terms):

1. \[ t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1) = \]
2. \[ t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2) = \]
3. \[ t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1) = \]

Example seq.2. If $s$ and $t$ are closed terms, then $s = t, \varphi(s) \vdash \varphi(t)$:

\[
\frac{\varphi(s) \Rightarrow \varphi(s)}{s = t, \varphi(s) \Rightarrow \varphi(s)}_{WL} = \\
\frac{s = t, \varphi(s) \Rightarrow \varphi(t)}{s = t, \varphi(s) \Rightarrow \varphi(t)}_{=}
\]

This may be familiar as the principle of substitutability of identicals, or Leibniz’ Law.

$LK$ proves that $=$ is symmetric and transitive:

\[
\frac{\Rightarrow t_1 = t_1}{t_1 = t_2 \Rightarrow t_1 = t_1} = \\
\frac{t_1 = t_2 \Rightarrow t_1 = t_1}{t_1 = t_2 \Rightarrow t_2 = t_1}_{WL} = \\
\frac{t_1 = t_2 \Rightarrow t_2 = t_1}{t_1 = t_2 \Rightarrow t_2 = t_1}_{WL} = \\
\frac{t_1 = t_2, t_1 = t_2 \Rightarrow t_1 = t_3}{t_2 = t_3, t_1 = t_2 \Rightarrow t_1 = t_3}_{XL} = \\
\frac{t_1 = t_2, t_2 = t_3 \Rightarrow t_1 = t_3}{t_1 = t_2, t_2 = t_3 \Rightarrow t_1 = t_3}_{XL} = \\
\frac{t_1 = t_2 \Rightarrow t_1 = t_2}{t_1 = t_2 \Rightarrow t_1 = t_2}_{XL} = \\
\frac{t_1 = t_2 \Rightarrow t_1 = t_2}{t_1 = t_2 \Rightarrow t_1 = t_2}_{XL}
\]

In the derivation on the left, the formula $x = t_1$ is our $\varphi(x)$. On the right, we take $\varphi(x)$ to be $t_1 = x$.

Problem seq.1. Give derivations of the following sequents:

1. $\Rightarrow \forall x \forall y ((x = y \land \varphi(x)) \rightarrow \varphi(y))$
2. $\exists x \varphi(x) \land \forall y \forall z ((\varphi(y) \land \varphi(z)) \rightarrow y = z) \Rightarrow \exists x (\varphi(x) \land \forall y (\varphi(y) \rightarrow y = x))$

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Bibliography