seq.1 Derivations

We’ve said what an initial sequent looks like, and we’ve given the rules of inference. Derivations in the sequent calculus are inductively generated from these: each derivation either is an initial sequent on its own, or consists of one or two derivations followed by an inference.

Definition seq.1 (LK derivation). An LK-derivation of a sequent \( S \) is a tree of sequents satisfying the following conditions:

1. The topmost sequents of the tree are initial sequents.
2. The bottommost sequent of the tree is \( S \).
3. Every sequent in the tree except \( S \) is a premise of a correct application of an inference rule whose conclusion stands directly below that sequent in the tree.

We then say that \( S \) is the end-sequent of the derivation and that \( S \) is derivable in LK (or LK-derivable).

Example seq.2. Every initial sequent, e.g., \( \chi \Rightarrow \chi \) is a derivation. We can obtain a new derivation from this by applying, say, the WL rule,

\[
\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \quad \text{WL}
\]

The rule, however, is meant to be general: we can replace the \( \varphi \) in the rule with any sentence, e.g., also with \( \theta \). If the premise matches our initial sequent \( \chi \Rightarrow \chi \), that means that both \( \Gamma \) and \( \Delta \) are just \( \chi \), and the conclusion would then be \( \theta, \chi \Rightarrow \chi \). So, the following is a derivation:

\[
\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \quad \text{WL}
\]

We can now apply another rule, say XL, which allows us to switch two sentences on the left. So, the following is also a correct derivation:

\[
\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \quad \text{WL}
\]

\[
\frac{\chi \Rightarrow \chi}{\chi, \theta \Rightarrow \chi} \quad \text{XL}
\]

In this application of the rule, which was given as

\[
\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \quad \text{XL}
\]

both \( \Gamma \) and \( \Pi \) were empty, \( \Delta \) is \( \chi \), and the roles of \( \varphi \) and \( \psi \) are played by \( \theta \) and \( \chi \), respectively. In much the same way, we also see that

\[
\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \quad \text{WL}
\]
is a derivation. Now we can take these two derivations, and combine them using \( \land R \). That rule was

\[
\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \land \psi} \land R
\]

In our case, the premises must match the last sequents of the derivations ending in the premises. That means that \( \Gamma \) is \( \chi, \theta \), \( \Delta \) is empty, \( \varphi \) is \( \chi \) and \( \psi \) is \( \theta \). So the conclusion, if the inference should be correct, is \( \chi, \theta \Rightarrow \chi \land \theta \).

\[
\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{ WL} \quad \frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \chi} \text{ WL} \\
\frac{\chi, \theta \Rightarrow \chi}{\chi, \theta \Rightarrow \chi} \text{ XL} \quad \frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{ WL} \\
\frac{\chi, \theta \Rightarrow \chi \land \theta}{\chi, \theta \Rightarrow \chi \land \theta} \land R
\]

Of course, we can also reverse the premises, then \( \varphi \) would be \( \theta \) and \( \psi \) would be \( \chi \).

\[
\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{ WL} \quad \frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{ WL} \\
\frac{\chi, \theta \Rightarrow \chi}{\chi, \theta \Rightarrow \chi} \text{ XL} \quad \frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \chi} \text{ WL} \\
\frac{\chi, \theta \Rightarrow \theta \land \chi}{\chi, \theta \Rightarrow \theta \land \chi} \land R
\]

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