seq.1 Derivations

fol:seq:der:sec We've said what an initial sequent looks like, and we've given the rules of inference. Derivations in the sequent calculus are inductively generated from these: each derivation either is an initial sequent on its own, or consists of one or two derivations followed by an inference. explanation

Definition seq.1 (LK derivation). An **LK**-*derivation* of a sequent S is a finite tree of sequents satisfying the following conditions:

1. The topmost sequents of the tree are initial sequents.
2. The bottommost sequent of the tree is S.
3. Every sequent in the tree except S is a premise of a correct application of an inference rule whose conclusion stands directly below that sequent in the tree.

We then say that S is the *end-sequent* of the derivation and that S is *derivable in* **LK** (or **LK**-derivable).

Example seq.2. Every initial sequent, e.g., $\chi \Rightarrow \chi$ is a derivation. We can obtain a new derivation from this by applying, say, the WL rule,

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ \text{WL}$$

The rule, however, is meant to be general: we can replace the φ in the rule with any sentence, e.g., also with θ. If the premise matches our initial sequent $\chi \Rightarrow \chi$, that means that both Γ and Δ are just χ, and the conclusion would then be $\theta, \chi \Rightarrow \chi$. So, the following is a derivation:

$$\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\ \text{WL}$$

We can now apply another rule, say XL, which allows us to switch two sentences on the left. So, the following is also a correct derivation:

$$\frac{\dfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\ \text{WL}}{\chi, \theta \Rightarrow \chi}\ \text{XL}$$

In this application of the rule, which was given as

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta,}\ \text{XL}$$

both Γ and Π were empty, Δ is χ, and the roles of φ and ψ are played by θ and χ, respectively. In much the same way, we also see that

$$\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta}\ \text{WL}$$

is a derivation. Now we can take these two derivations, and combine them using $\wedge$R. That rule was

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\wedge\text{R}$$

In our case, the premises must match the last sequents of the derivations ending in the premises. That means that Γ is χ, θ, Δ is empty, φ is χ and ψ is θ. So the conclusion, if the inference should be correct, is $\chi, \theta \Rightarrow \chi \wedge \theta$.

$$\dfrac{\dfrac{\dfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\text{WL}}{\chi, \theta \Rightarrow \chi}\text{XL} \qquad \dfrac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta}\text{WL}}{\chi, \theta \Rightarrow \chi \wedge \theta}\wedge\text{R}$$

Of course, we can also reverse the premises, then φ would be θ and ψ would be χ.

$$\dfrac{\dfrac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta}\text{WL} \qquad \dfrac{\dfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\text{WL}}{\chi, \theta \Rightarrow \chi}\text{XL}}{\chi, \theta \Rightarrow \theta \wedge \chi}\wedge\text{R}$$

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Bibliography