seq.1 Derivations

We've said what an initial sequent looks like, and we've given the rules of inference. Derivations in the sequent calculus are inductively generated from these: each derivation either is an initial sequent on its own, or consists of one or two derivations followed by an inference.

Definition seq.1 (LK derivation). An LK-derivation of a sequent $S$ is a tree of sequents satisfying the following conditions:

1. The topmost sequents of the tree are initial sequents.
2. The bottommost sequent of the tree is $S$.
3. Every sequent in the tree except $S$ is a premise of a correct application of an inference rule whose conclusion stands directly below that sequent in the tree.

We then say that $S$ is the end-sequent of the derivation and that $S$ is derivable in LK (or LK-derivable).

Example seq.2. Every initial sequent, e.g., $\chi \Rightarrow \chi$ is a derivation. We can obtain a new derivation from this by applying, say, the WL rule,

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{ WL}$$

The rule, however, is meant to be general: we can replace the $\varphi$ in the rule with any sentence, e.g., also with $\theta$. If the premise matches our initial sequent $\chi \Rightarrow \chi$, that means that both $\Gamma$ and $\Delta$ are just $\chi$, and the conclusion would then be $\theta, \chi \Rightarrow \chi$. So, the following is a derivation:

$$\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{ WL}$$

We can now apply another rule, say XL, which allows us to switch two sentences on the left. So, the following is also a correct derivation:

$$\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{ WL}$$
$$\frac{\chi, \theta \Rightarrow \chi}{\chi, \theta \Rightarrow \chi} \text{ XL}$$

In this application of the rule, which was given as

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \text{ XL}$$

both $\Gamma$ and $\Pi$ were empty, $\Delta$ is $\chi$, and the roles of $\varphi$ and $\psi$ are played by $\theta$ and $\chi$, respectively. In much the same way, we also see that

$$\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{ WL}$$
is a derivation. Now we can take these two derivations, and combine them using $\land R$. That rule was

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \land \psi} \land R$$

In our case, the premises must match the last sequents of the derivations ending in the premises. That means that $\Gamma$ is $\chi, \theta$, $\Delta$ is empty, $\varphi$ is $\chi$ and $\psi$ is $\theta$. So the conclusion, if the inference should be correct, is $\chi, \theta \Rightarrow \chi \land \theta$. Of course, we can also reverse the premises, then $\varphi$ would be $\theta$ and $\psi$ would be $\chi$. So both of the following are correct derivations.

$$\chi \Rightarrow \chi \quad WL$$

$$\begin{align*}
\frac{\theta, \chi \Rightarrow \chi}{\chi, \theta \Rightarrow \chi} & \quad XL
\frac{\chi, \theta \Rightarrow \chi}{\chi, \theta \Rightarrow \chi \land \theta} & \quad \land R
\frac{\chi \Rightarrow \chi}{\theta \Rightarrow \theta} & \quad WL
\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \chi \land \theta} & \quad \land R
\end{align*}$$

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Bibliography