

## seq.1 Derivations

fol:seq:der: We've said what an initial sequent looks like, and we've given the rules of explanation  
 sec inference. **Derivations** in the sequent calculus are inductively generated from these: each **derivation** either is an initial sequent on its own, or consists of one or two **derivations** followed by an inference.

**Definition seq.1 (LK derivation).** An **LK-derivation** of a sequent  $S$  is a tree of sequents satisfying the following conditions:

1. The topmost sequents of the tree are initial sequents.
2. The bottommost sequent of the tree is  $S$ .
3. Every sequent in the tree except  $S$  is a premise of a correct application of an inference rule whose conclusion stands directly below that sequent in the tree.

We then say that  $S$  is the *end-sequent* of the **derivation** and that  $S$  is *derivable in LK* (or **LK-derivable**).

**Example seq.2.** Every initial sequent, e.g.,  $\chi \Rightarrow \chi$  is a **derivation**. We can obtain a new **derivation** from this by applying, say, the WL rule,

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{WL}$$

The rule, however, is meant to be general: we can replace the  $\varphi$  in the rule with any **sentence**, e.g., also with  $\theta$ . If the premise matches our initial sequent  $\chi \Rightarrow \chi$ , that means that both  $\Gamma$  and  $\Delta$  are just  $\chi$ , and the conclusion would then be  $\theta, \chi \Rightarrow \chi$ . So, the following is a **derivation**:

$$\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}$$

We can now apply another rule, say XL, which allows us to switch two **sentences** on the left. So, the following is also a correct **derivation**:

$$\frac{\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}}{\chi, \theta \Rightarrow \chi} \text{XL}$$

In this application of the rule, which was given as

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \text{XL}$$

both  $\Gamma$  and  $\Pi$  were empty,  $\Delta$  is  $\chi$ , and the roles of  $\varphi$  and  $\psi$  are played by  $\theta$  and  $\chi$ , respectively. In much the same way, we also see that

$$\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{WL}$$

is a **derivation**. Now we can take these two derivations, and combine them using  $\wedge R$ . That rule was

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge R$$

In our case, the premises must match the last sequents of the **derivations** ending in the premises. That means that  $\Gamma$  is  $\chi, \theta$ ,  $\Delta$  is empty,  $\varphi$  is  $\chi$  and  $\psi$  is  $\theta$ . So the conclusion, if the inference should be correct, is  $\chi, \theta \Rightarrow \chi \wedge \theta$ . Of course, we can also reverse the premises, then  $\varphi$  would be  $\theta$  and  $\psi$  would be  $\chi$ . So both of the following are correct **derivations**.

$$\frac{\frac{\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}}{\chi, \theta \Rightarrow \chi} \text{XL} \quad \frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{WL}}{\chi, \theta \Rightarrow \chi \wedge \theta} \wedge R \quad \frac{\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{WL} \quad \frac{\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}}{\chi, \theta \Rightarrow \chi} \text{XL}}{\chi, \theta \Rightarrow \theta \wedge \chi} \wedge R$$

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## Bibliography