

seq.1 Derivations

fol:seq:der: We've said what an initial sequent looks like, and we've given the rules of explanation
 sec inference. **Derivations** in the sequent calculus are inductively generated from these: each **derivation** either is an initial sequent on its own, or consists of one or two **derivations** followed by an inference.

Definition seq.1 (LK derivation). An **LK-derivation** of a sequent S is a tree of sequents satisfying the following conditions:

1. The topmost sequents of the tree are initial sequents.
2. The bottommost sequent of the tree is S .
3. Every sequent in the tree except S is a premise of a correct application of an inference rule whose conclusion stands directly below that sequent in the tree.

We then say that S is the *end-sequent* of the **derivation** and that S is *derivable in LK* (or **LK-derivable**).

Example seq.2. Every initial sequent, e.g., $\chi \Rightarrow \chi$ is a **derivation**. We can obtain a new **derivation** from this by applying, say, the WL rule,

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{WL}$$

The rule, however, is meant to be general: we can replace the φ in the rule with any **sentence**, e.g., also with θ . If the premise matches our initial sequent $\chi \Rightarrow \chi$, that means that both Γ and Δ are just χ , and the conclusion would then be $\theta, \chi \Rightarrow \chi$. So, the following is a **derivation**:

$$\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}$$

We can now apply another rule, say XL, which allows us to switch two **sentences** on the left. So, the following is also a correct **derivation**:

$$\frac{\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}}{\chi, \theta \Rightarrow \chi} \text{XL}$$

In this application of the rule, which was given as

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \text{XL}$$

both Γ and Π were empty, Δ is χ , and the roles of φ and ψ are played by θ and χ , respectively. In much the same way, we also see that

$$\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{WL}$$

is a **derivation**. Now we can take these two derivations, and combine them using $\wedge R$. That rule was

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge R$$

In our case, the premises must match the last sequents of the **derivations** ending in the premises. That means that Γ is χ, θ , Δ is empty, φ is χ and ψ is θ . So the conclusion, if the inference should be correct, is $\chi, \theta \Rightarrow \chi \wedge \theta$. Of course, we can also reverse the premises, then φ would be θ and ψ would be χ . So both of the following are correct **derivations**.

$$\frac{\frac{\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}}{\chi, \theta \Rightarrow \chi} \text{XL} \quad \frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{WL}}{\chi, \theta \Rightarrow \chi \wedge \theta} \wedge R \quad \frac{\frac{\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}}{\chi, \theta \Rightarrow \theta} \text{WL} \quad \frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \chi} \text{XL}}{\chi, \theta \Rightarrow \theta \wedge \chi} \wedge R$$

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Bibliography