prf.1 The Sequent Calculus

While many derivation systems operate with arrangements of sentences, the sequent calculus operates with sequents. A sequent is an expression of the form

\[ \varphi_1, \ldots, \varphi_m \Rightarrow \psi_1, \ldots, \psi_m, \]

that is a pair of sequences of sentences, separated by the sequent symbol \( \Rightarrow \). Either sequence may be empty. A derivation in the sequent calculus is a tree of sequents, where the topmost sequents are of a special form (they are called “initial sequents” or “axioms”) and every other sequent follows from the sequents immediately above it by one of the rules of inference. The rules of inference either manipulate the sentences in the sequents (adding, removing, or rearranging them on either the left or the right), or they introduce a complex formula in the conclusion of the rule. For instance, the \( \land L \) rule allows the inference from \( \varphi, \Gamma \Rightarrow \Delta \) to \( \varphi \land \psi, \Gamma \Rightarrow \Delta \), and the \( \rightarrow R \) allows the inference from \( \varphi, \Gamma \Rightarrow \Delta, \psi \) to \( \Gamma \Rightarrow \Delta, \varphi \rightarrow \psi \), for any \( \Gamma, \Delta, \varphi, \psi \). (In particular, \( \Gamma \) and \( \Delta \) may be empty.)

The \( \vdash \) relation based on the sequent calculus is defined as follows: \( \Gamma \vdash \varphi \) iff there is some sequence \( \Gamma_0 \) such that every \( \varphi \) in \( \Gamma_0 \) is in \( \Gamma \) and there is a derivation with the sequent \( \Gamma_0 \Rightarrow \varphi \) at its root. \( \varphi \) is a theorem in the sequent calculus if the sequent \( \Rightarrow \varphi \) has a derivation. For instance, here is a derivation that shows that \( \vdash (\varphi \land \psi) \rightarrow \varphi \):

\[
\begin{align*}
\varphi & \Rightarrow \varphi \quad & \text{\( \land L \) rule} \\
\varphi \land \psi & \Rightarrow \varphi \quad & \text{\( \rightarrow R \) rule} \\
\Rightarrow (\varphi \land \psi) & \rightarrow \varphi
\end{align*}
\]

A set \( \Gamma \) is inconsistent in the sequent calculus if there is a derivation of \( \Gamma \Rightarrow \) (where every \( \varphi \in \Gamma_0 \) is in \( \Gamma \) and the right side of the sequent is empty). Using the rule WR, any sentence can be derived from an inconsistent set.

The sequent calculus was invented in the 1930s by Gerhard Gentzen. Because of its systematic and symmetric design, it is a very useful formalism for developing a theory of derivations. It is relatively easy to find derivations in the sequent calculus, but these derivations are often hard to read and their connection to proofs are sometimes not easy to see. It has proved to be a very elegant approach to derivation systems, however, and many logics have sequent calculus systems.

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Bibliography