

prf.1 Introduction

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Logics commonly have both a semantics and a **derivation** system. The semantics concerns concepts such as truth, satisfiability, validity, and entailment. The purpose of **derivation** systems is to provide a purely syntactic method of establishing entailment and validity. They are purely syntactic in the sense that a **derivation** in such a system is a finite syntactic object, usually a sequence (or other finite arrangement) of **sentences** or **formulas**. Good **derivation** systems have the property that any given sequence or arrangement of **sentences** or **formulas** can be verified mechanically to be “correct.”

The simplest (and historically first) **derivation** systems for first-order logic were *axiomatic*. A sequence of **formulas** counts as a **derivation** in such a system if each individual **formula** in it is either among a fixed set of “axioms” or follows from **formulas** coming before it in the sequence by one of a fixed number of “inference rules”—and it can be mechanically verified if a **formula** is an axiom and whether it follows correctly from other **formulas** by one of the inference rules. Axiomatic proof systems are easy to describe—and also easy to handle meta-theoretically—but **derivations** in them are hard to read and understand, and are also hard to produce.

Other **derivation** systems have been developed with the aim of making it easier to construct **derivations** or easier to understand **derivations** once they are complete. Examples are natural deduction, truth trees, also known as tableaux proofs, and the sequent calculus. Some **derivation** systems are designed especially with mechanization in mind, e.g., the resolution method is easy to implement in software (but its **derivations** are essentially impossible to understand). Most of these other proof systems represent **derivations** as trees of **formulas** rather than sequences. This makes it easier to see which parts of a **derivation** depend on which other parts.

So for a given logic, such as first-order logic, the different **derivation** systems will give different explications of what it is for a **sentence** to be a *theorem* and what it means for a **sentence** to be **derivable** from some others. However that is done (via axiomatic **derivations**, natural deductions, sequent **derivations**, truth trees, resolution refutations), we want these relations to match the semantic notions of validity and entailment. Let’s write $\vdash \varphi$ for “ φ is a theorem” and “ $\Gamma \vdash \varphi$ ” for “ φ is **derivable** from Γ .” However \vdash is defined, we want it to match up with \models , that is:

1. $\vdash \varphi$ if and only if $\models \varphi$
2. $\Gamma \vdash \varphi$ if and only if $\Gamma \models \varphi$

The “only if” direction of the above is called *soundness*. A **derivation** system is sound if **derivability** guarantees entailment (or validity). Every decent **derivation** system has to be sound; unsound **derivation** systems are not useful at all. After all, the entire purpose of a **derivation** is to provide a syntactic guarantee of validity or entailment. We’ll prove soundness for the **derivation** systems we present.

The converse “if” direction is also important: it is called *completeness*. A complete **derivation** system is strong enough to show that φ is a theorem whenever φ is valid, and that there $\Gamma \vdash \varphi$ whenever $\Gamma \models \varphi$. Completeness is harder to establish, and some logics have no complete **derivation** systems. First-order logic does. Kurt Gödel was the first one to prove completeness for a **derivation** system of first-order logic in his 1929 dissertation.

Another concept that is connected to **derivation** systems is that of *consistency*. A set of **sentences** is called inconsistent if anything whatsoever can be **derived** from it, and consistent otherwise. Inconsistency is the syntactic counterpart to unsatisfiability: like unsatisfiable sets, inconsistent sets of **sentences** do not make good theories, they are defective in a fundamental way. Consistent sets of **sentences** may not be true or useful, but at least they pass that minimal threshold of logical usefulness. For different **derivation** systems the specific definition of consistency of sets of **sentences** might differ, but like \vdash , we want consistency to coincide with its semantic counterpart, satisfiability. We want it to always be the case that Γ is consistent if and only if it is satisfiable. Here, the “if” direction amounts to completeness (consistency guarantees satisfiability), and the “only if” direction amounts to soundness (satisfiability guarantees consistency). In fact, for classical first-order logic, the two versions of soundness and completeness are equivalent.

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Bibliography