

## prf.1 Axiomatic Derivations

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sec Axiomatic **derivations** are the oldest and simplest logical **derivation** systems. Its **derivations** are simply sequences of **sentences**. A sequence of **sentences** counts as a correct **derivation** if every **sentence**  $\varphi$  in it satisfies one of the following conditions:

1.  $\varphi$  is an axiom, or
2.  $\varphi$  is an **element** of a given set  $\Gamma$  of **sentences**, or
3.  $\varphi$  is justified by a rule of inference.

To be an axiom,  $\varphi$  has to have the form of one of a number of fixed **sentence** schemas. There are many sets of axiom schemas that provide a satisfactory (sound and complete) **derivation** system for first-order logic. Some are organized according to the connectives they govern, e.g., the schemas

$$\varphi \rightarrow (\psi \rightarrow \varphi) \quad \psi \rightarrow (\psi \vee \chi) \quad (\psi \wedge \chi) \rightarrow \psi$$

are common axioms that govern  $\rightarrow$ ,  $\vee$  and  $\wedge$ . Some axiom systems aim at a minimal number of axioms. Depending on the connectives that are taken as primitives, it is even possible to find axiom systems that consist of a single axiom.

A rule of inference is a conditional statement that gives a sufficient condition for a **sentence** in a **derivation** to be justified. Modus ponens is one very common such rule: it says that if  $\varphi$  and  $\varphi \rightarrow \psi$  are already justified, then  $\psi$  is justified. This means that a line in a **derivation** containing the **sentence**  $\psi$  is justified, provided that both  $\varphi$  and  $\varphi \rightarrow \psi$  (for some **sentence**  $\varphi$ ) appear in the **derivation** before  $\psi$ .

The  $\vdash$  relation based on axiomatic **derivations** is defined as follows:  $\Gamma \vdash \varphi$  iff there is a **derivation** with the **sentence**  $\varphi$  as its last formula (and  $\Gamma$  is taken as the set of **sentences** in that derivation which are justified by (2) above).  $\varphi$  is a theorem if  $\varphi$  has a **derivation** where  $\Gamma$  is empty, i.e., every **sentence** in the derivation is justified either by (1) or (3). For instance, here is a **derivation** that shows that  $\vdash \varphi \rightarrow (\psi \rightarrow (\psi \vee \varphi))$ :

1.  $\psi \rightarrow (\psi \vee \varphi)$
2.  $(\psi \rightarrow (\psi \vee \varphi)) \rightarrow (\varphi \rightarrow (\psi \rightarrow (\psi \vee \varphi)))$
3.  $\varphi \rightarrow (\psi \rightarrow (\psi \vee \varphi))$

The **sentence** on line 1 is of the form of the axiom  $\varphi \rightarrow (\varphi \vee \psi)$  (with the roles of  $\varphi$  and  $\psi$  reversed). The sentence on line 2 is of the form of the axiom  $\varphi \rightarrow (\psi \rightarrow \varphi)$ . Thus, both lines are justified. Line 3 is justified by modus ponens: if we abbreviate it as  $\theta$ , then line 2 has the form  $\chi \rightarrow \theta$ , where  $\chi$  is  $\psi \rightarrow (\psi \vee \varphi)$ , i.e., line 1.

A set  $\Gamma$  is inconsistent if  $\Gamma \vdash \perp$ . A complete axiom system will also prove that  $\perp \rightarrow \varphi$  for any  $\varphi$ , and so if  $\Gamma$  is inconsistent, then  $\Gamma \vdash \varphi$  for any  $\varphi$ .

Systems of axiomatic **derivations** for logic were first given by Gottlob Frege in his 1879 *Begriffsschrift*, which for this reason is often considered the first work of modern logic. They were perfected in Alfred North Whitehead and Bertrand Russell's *Principia Mathematica* and by David Hilbert and his students in the 1920s. They are thus often called "Frege systems" or "Hilbert systems." They are very versatile in that it is often easy to find an axiomatic system for a logic. Because **derivations** have a very simple structure and only one or two inference rules, it is also relatively easy to prove things *about* them. However, they are very hard to use in practice, i.e., it is difficult to find and write proofs.

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## **Bibliography**