

## ntd.1 Soundness

fol:ntd:sou:  
sec A **derivation** system, such as natural deduction, is *sound* if it cannot **derive** explanation things that do not actually follow. Soundness is thus a kind of guaranteed safety property for **derivation** systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

1. every **derivable sentence** is valid;
2. if a **sentence** is **derivable** from some others, it is also a consequence of them;
3. if a set of **sentences** is inconsistent, it is unsatisfiable.

These are important properties of a **derivation** system. If any of them do not hold, the **derivation** system is deficient—it would **derive** too much. Consequently, establishing the soundness of a **derivation** system is of the utmost importance.

fol:ntd:sou:  
thm:soundness **Theorem ntd.1** (Soundness). *If  $\varphi$  is **derivable** from the **undischarged** assumptions  $\Gamma$ , then  $\Gamma \models \varphi$ .*

*Proof.* Let  $\delta$  be a **derivation** of  $\varphi$ . We proceed by induction on the number of inferences in  $\delta$ .

For the induction basis we show the claim if the number of inferences is 0. In this case,  $\delta$  consists only of an initial **formula**. Every initial **formula**  $\varphi$  is an **undischarged** assumption, and as such, any **structure**  $\mathfrak{M}$  that satisfies all of the **undischarged** assumptions of the proof also satisfies  $\varphi$ .

Now for the inductive step. Suppose that  $\delta$  contains  $n$  inferences. The premise(s) of the lowermost inference are **derived** using sub-**derivations**, each of which contains fewer than  $n$  inferences. We assume the induction hypothesis: The premises of the last inference follow from the **undischarged** assumptions of the sub-**derivations** ending in those premises. We have to show that  $\varphi$  follows from the **undischarged** assumptions of the entire proof.

We distinguish cases according to the type of the lowermost inference. First, we consider the possible inferences with only one premise.

1. Suppose that the last inference is  $\neg$ -Intro: The **derivation** has the form

$$\begin{array}{c} \Gamma, [\varphi]^n \\ \vdots \\ \delta_1 \\ \vdots \\ \perp \\ \hline \neg\varphi \quad \neg\text{-Intro} \end{array}$$

By inductive hypothesis,  $\perp$  follows from the **undischarged** assumptions  $\Gamma \cup \{\varphi\}$  of  $\delta_1$ . Consider a **structure**  $\mathfrak{M}$ . We need to show that, if  $\mathfrak{M} \models \Gamma$ , then  $\mathfrak{M} \models \neg\varphi$ . Suppose for reductio that  $\mathfrak{M} \models \Gamma$ , but  $\mathfrak{M} \not\models \neg\varphi$ , i.e.,

$\mathfrak{M} \models \varphi$ . This would mean that  $\mathfrak{M} \models \Gamma \cup \{\varphi\}$ . This is contrary to our inductive hypothesis. So,  $\mathfrak{M} \models \neg\varphi$ .

2. The last inference is  $\wedge$ Elim: There are two variants:  $\varphi$  or  $\psi$  may be inferred from the premise  $\varphi \wedge \psi$ . Consider the first case. The derivation  $\delta$  looks like this:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \delta_1 \\ \vdots \\ \varphi \wedge \psi \end{array}}{\varphi} \wedge\text{Elim}$$

By inductive hypothesis,  $\varphi \wedge \psi$  follows from the **undischarged** assumptions  $\Gamma$  of  $\delta_1$ . Consider a **structure**  $\mathfrak{M}$ . We need to show that, if  $\mathfrak{M} \models \Gamma$ , then  $\mathfrak{M} \models \varphi$ . Suppose  $\mathfrak{M} \models \Gamma$ . By our inductive hypothesis ( $\Gamma \models \varphi \vee \psi$ ), we know that  $\mathfrak{M} \models \varphi \wedge \psi$ . By definition,  $\mathfrak{M} \models \varphi \wedge \psi$  iff  $\mathfrak{M} \models \varphi$  and  $\mathfrak{M} \models \psi$ . (The case where  $\psi$  is inferred from  $\varphi \wedge \psi$  is handled similarly.)

3. The last inference is  $\vee$ Intro: There are two variants:  $\varphi \vee \psi$  may be inferred from the premise  $\varphi$  or the premise  $\psi$ . Consider the first case. The derivation has the form

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \delta_1 \\ \vdots \\ \varphi \end{array}}{\varphi \vee \psi} \vee\text{Intro}$$

By inductive hypothesis,  $\varphi$  follows from the **undischarged** assumptions  $\Gamma$  of  $\delta_1$ . Consider a **structure**  $\mathfrak{M}$ . We need to show that, if  $\mathfrak{M} \models \Gamma$ , then  $\mathfrak{M} \models \varphi \vee \psi$ . Suppose  $\mathfrak{M} \models \Gamma$ ; then  $\mathfrak{M} \models \varphi$  since  $\Gamma \models \varphi$  (the inductive hypothesis). So it must also be the case that  $\mathfrak{M} \models \varphi \vee \psi$ . (The case where  $\varphi \vee \psi$  is inferred from  $\psi$  is handled similarly.)

4. The last inference is  $\rightarrow$ Intro:  $\varphi \rightarrow \psi$  is inferred from a subproof with assumption  $\varphi$  and conclusion  $\psi$ , i.e.,

$$\frac{\begin{array}{c} \Gamma, [\varphi]^n \\ \vdots \\ \delta_1 \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \rightarrow\text{Intro}$$

By inductive hypothesis,  $\psi$  follows from the **undischarged** assumptions of  $\delta_1$ , i.e.,  $\Gamma \cup \{\varphi\} \models \psi$ . Consider a **structure**  $\mathfrak{M}$ . The **undischarged** assumptions of  $\delta$  are just  $\Gamma$ , since  $\varphi$  is discharged at the last inference. So we need to show that  $\Gamma \models \varphi \rightarrow \psi$ . For reductio, suppose that for some **structure**  $\mathfrak{M}$ ,  $\mathfrak{M} \models \Gamma$  but  $\mathfrak{M} \not\models \varphi \rightarrow \psi$ . So,  $\mathfrak{M} \models \varphi$  and  $\mathfrak{M} \not\models \psi$ . But by hypothesis,  $\psi$  is a consequence of  $\Gamma \cup \{\varphi\}$ , i.e.,  $\mathfrak{M} \models \psi$ , which is a contradiction. So,  $\Gamma \models \varphi \rightarrow \psi$ .

5. The last inference is  $\perp_I$ : Here,  $\delta$  ends in

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \delta_1 \\ \vdots \\ \perp \end{array}}{\varphi} \perp_I$$

By induction hypothesis,  $\Gamma \models \perp$ . We have to show that  $\Gamma \models \varphi$ . Suppose not; then for some  $\mathfrak{M}$  we have  $\mathfrak{M} \models \Gamma$  and  $\mathfrak{M} \not\models \varphi$ . But we always have  $\mathfrak{M} \models \perp$ , so this would mean that  $\Gamma \not\models \perp$ , contrary to the induction hypothesis.

6. The last inference is  $\perp_C$ : Exercise.  
 7. The last inference is  $\forall$ Intro: Then  $\delta$  has the form

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \delta_1 \\ \vdots \\ \varphi(a) \end{array}}{\forall x \varphi(x)} \forall\text{Intro}$$

The premise  $\varphi(a)$  is a consequence of the **undischarged** assumptions  $\Gamma$  by induction hypothesis. Consider some structure,  $\mathfrak{M}$ , such that  $\mathfrak{M} \models \Gamma$ . We need to show that  $\mathfrak{M} \models \forall x \varphi(x)$ . Since  $\forall x \varphi(x)$  is a **sentence**, this means we have to show that for every variable assignment  $s$ ,  $\mathfrak{M}, s \models \varphi(x)$  (??). Since  $\Gamma$  consists entirely of sentences,  $\mathfrak{M}, s \models \psi$  for all  $\psi \in \Gamma$  by ???. Let  $\mathfrak{M}'$  be like  $\mathfrak{M}$  except that  $a^{\mathfrak{M}'} = s(x)$ . Since  $a$  does not occur in  $\Gamma$ ,  $\mathfrak{M}' \models \Gamma$  by ???. Since  $\Gamma \models A(a)$ ,  $\mathfrak{M}' \models A(a)$ . Since  $\varphi(a)$  is a **sentence**,  $\mathfrak{M}, s \models \varphi(a)$  by ???.  $\mathfrak{M}', s \models \varphi(x)$  iff  $\mathfrak{M}' \models \varphi(a)$  by ?? (recall that  $\varphi(a)$  is just  $\varphi(x)[a/x]$ ). So,  $\mathfrak{M}', s \models \varphi(x)$ . Since  $a$  does not occur in  $\varphi(x)$ , by ??,  $\mathfrak{M}, s \models \varphi(x)$ . But  $s$  was an arbitrary variable assignment, so  $\mathfrak{M} \models \forall x \varphi(x)$ .

8. The last inference is  $\exists$ Intro: Exercise.  
 9. The last inference is  $\forall$ Elim: Exercise.

Now let's consider the possible inferences with several premises:  $\forall$ Elim,  $\wedge$ Intro,  $\rightarrow$ Elim, and  $\exists$ Elim.

1. The last inference is  $\wedge$ Intro.  $\varphi \wedge \psi$  is inferred from the premises  $\varphi$  and  $\psi$  and  $\delta$  has the form

$$\frac{\begin{array}{c} \Gamma_1 \\ \vdots \\ \delta_1 \\ \vdots \\ \varphi \end{array} \quad \begin{array}{c} \Gamma_2 \\ \vdots \\ \delta_2 \\ \vdots \\ \psi \end{array}}{\varphi \wedge \psi} \wedge\text{Intro}$$

By induction hypothesis,  $\varphi$  follows from the **undischarged** assumptions  $\Gamma_1$  of  $\delta_1$  and  $\psi$  follows from the **undischarged** assumptions  $\Gamma_2$  of  $\delta_2$ . The **undischarged** assumptions of  $\delta$  are  $\Gamma_1 \cup \Gamma_2$ , so we have to show that  $\Gamma_1 \cup \Gamma_2 \models \varphi \wedge \psi$ . Consider a **structure**  $\mathfrak{M}$  with  $\mathfrak{M} \models \Gamma_1 \cup \Gamma_2$ . Since  $\mathfrak{M} \models \Gamma_1$ , it must be the case that  $\mathfrak{M} \models \varphi$  as  $\Gamma_1 \models \varphi$ , and since  $\mathfrak{M} \models \Gamma_2$ ,  $\mathfrak{M} \models \psi$  since  $\Gamma_2 \models \psi$ . Together,  $\mathfrak{M} \models \varphi \wedge \psi$ .

2. The last inference is  $\forall$ Elim: Exercise.
3. The last inference is  $\rightarrow$ Elim.  $\psi$  is inferred from the premises  $\varphi \rightarrow \psi$  and  $\varphi$ . The derivation  $\delta$  looks like this:

$$\frac{\begin{array}{c} \Gamma_1 \\ \vdots \\ \delta_1 \\ \vdots \\ \varphi \rightarrow \psi \end{array} \quad \begin{array}{c} \Gamma_2 \\ \vdots \\ \delta_2 \\ \vdots \\ \varphi \end{array}}{\psi} \rightarrow\text{Elim}$$

By induction hypothesis,  $\varphi \rightarrow \psi$  follows from the **undischarged** assumptions  $\Gamma_1$  of  $\delta_1$  and  $\varphi$  follows from the **undischarged** assumptions  $\Gamma_2$  of  $\delta_2$ . Consider a **structure**  $\mathfrak{M}$ . We need to show that, if  $\mathfrak{M} \models \Gamma_1 \cup \Gamma_2$ , then  $\mathfrak{M} \models \psi$ . Suppose  $\mathfrak{M} \models \Gamma_1 \cup \Gamma_2$ . Since  $\Gamma_1 \models \varphi \rightarrow \psi$ ,  $\mathfrak{M} \models \varphi \rightarrow \psi$ . Since  $\Gamma_2 \models \varphi$ , we have  $\mathfrak{M} \models \varphi$ . This means that  $\mathfrak{M} \models \psi$  (For if  $\mathfrak{M} \not\models \psi$ , since  $\mathfrak{M} \models \varphi$ , we'd have  $\mathfrak{M} \not\models \varphi \rightarrow \psi$ , contradicting  $\mathfrak{M} \models \varphi \rightarrow \psi$ ).

4. The last inference is  $\neg$ Elim: Exercise.
5. The last inference is  $\exists$ Elim: Exercise.

□

**Problem ntd.1.** Complete the proof of [Theorem ntd.1](#).

**Corollary ntd.2.** *If  $\vdash \varphi$ , then  $\varphi$  is valid.*

[fol:ntd:sou:](#)  
[cor:weak-soundness](#)

**Corollary ntd.3.** *If  $\Gamma$  is satisfiable, then it is consistent.*

[fol:ntd:sou:](#)  
[cor:consistency-soundness](#)

*Proof.* We prove the contrapositive. Suppose that  $\Gamma$  is not consistent. Then  $\Gamma \vdash \perp$ , i.e., there is a **derivation** of  $\perp$  from **undischarged** assumptions in  $\Gamma$ . By [Theorem ntd.1](#), any **structure**  $\mathfrak{M}$  that satisfies  $\Gamma$  must satisfy  $\perp$ . Since  $\mathfrak{M} \not\models \perp$  for every **structure**  $\mathfrak{M}$ , no  $\mathfrak{M}$  can satisfy  $\Gamma$ , i.e.,  $\Gamma$  is not satisfiable.  $\square$

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## Bibliography