ntd.1 Soundness with Identity predicate

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Proposition ntd.1. *Natural deduction with rules for $=$ is sound.*

Proof. Any formula of the form $t = t$ is valid, since for every structure $\mathfrak{M}$, $\mathfrak{M} \vDash t = t$. (Note that we assume the term t to be closed, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a derivation is $=$Elim, i.e., the derivation has the following form:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\ \delta_1\\ t_1 = t_2\end{array} \qquad \begin{array}{c}\Gamma_2\\ \vdots\ \delta_2\\ \varphi(t_1)\end{array}}{\varphi(t_2)}\ {=}\mathrm{Elim}$$

The premises $t_1 = t_2$ and $\varphi(t_1)$ are derived from undischarged assumptions Γ_1 and Γ_2, respectively. We want to show that $\varphi(t_2)$ follows from $\Gamma_1 \cup \Gamma_2$. Consider a structure $\mathfrak{M}$ with $\mathfrak{M} \vDash \Gamma_1 \cup \Gamma_2$. By induction hypothesis, $\mathfrak{M} \vDash \varphi(t_1)$ and $\mathfrak{M} \vDash t_1 = t_2$. Therefore, $\mathrm{Val}^{\mathfrak{M}}(t_1) = \mathrm{Val}^{\mathfrak{M}}(t_2)$. Let s be any variable assignment, and $m = \mathrm{Val}^{\mathfrak{M}}(t_1) = \mathrm{Val}^{\mathfrak{M}}(t_2)$. By **??**, $\mathfrak{M}, s \vDash \varphi(t_1)$ iff $\mathfrak{M}, s[m/x] \vDash \varphi(x)$ iff $\mathfrak{M}, s \vDash \varphi(t_2)$. Since $\mathfrak{M} \vDash \varphi(t_1)$, we have $\mathfrak{M} \vDash \varphi(t_2)$. □

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Bibliography