Proposition ntd.1. Natural deduction with rules for $=$ is sound.

Proof. Any formula of the form $t = t$ is valid, since for every structure $\mathcal{M}$, $\mathcal{M} \models t = t$. (Note that we assume the term $t$ to be closed, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a derivation is $=\text{Elim}$, i.e., the derivation has the following form:

$$
\begin{array}{c}
\Gamma_1 \\
\vdots \\
\delta_1 \\
\vdots \\
t_1 = t_2 \\
\vdots \\
\delta_2 \\
\vdots \\
\varphi(t_1) \\
\end{array}
\Rightarrow
\begin{array}{c}
\vdots \\
\vdots \\
\varphi(t_2) \\
\end{array}
=\text{Elim}
$$

The premises $t_1 = t_2$ and $\varphi(t_1)$ are derived from undischarged assumptions $\Gamma_1$ and $\Gamma_2$, respectively. We want to show that $\varphi(t_2)$ follows from $\Gamma_1 \cup \Gamma_2$. Consider a structure $\mathcal{M}$ with $\mathcal{M} \models \Gamma_1 \cup \Gamma_2$. By induction hypothesis, $\mathcal{M} \models \varphi(t_1)$ and $\mathcal{M} \models t_1 = t_2$. Therefore, $\text{Val}_\mathcal{M}(t_1) = \text{Val}_\mathcal{M}(t_2)$. Let $s$ be any variable assignment, and $m = \text{Val}_\mathcal{M}(t_1) = \text{Val}_\mathcal{M}(t_2)$. By $??$, $\mathcal{M}, s \models \varphi(t_1)$ iff $\mathcal{M}, s[m/x] \models \varphi(x)$ iff $\mathcal{M}, s \models \varphi(t_2)$. Since $\mathcal{M} \models \varphi(t_1)$, we have $\mathcal{M} \models \varphi(t_2)$. $\square$

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Bibliography