ntd.1 Soundness with Identity predicate

Proposition ntd.1. Natural deduction with rules for = is sound.

Proof. Any formula of the form \( t = t \) is valid, since for every structure \( \mathcal{M} \), \( \mathcal{M} \models t = t \). (Note that we assume the term \( t \) to be ground, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a derivation is \( =\text{Elim} \), i.e., the derivation has the following form:

\[
\begin{array}{c}
\Gamma_1 \\
\vdots \\
\delta_1 \\
\vdots \\
\delta_2 \\
\vdots \\
t_1 = t_2 \\
\varphi(t_1) \\
\hline
\varphi(t_2)
\end{array}
\text{=Elim}
\]

The premises \( t_1 = t_2 \) and \( \varphi(t_1) \) are derived from undischarged assumptions \( \Gamma_1 \) and \( \Gamma_2 \), respectively. We want to show that \( \varphi(t_2) \) follows from \( \Gamma_1 \cup \Gamma_2 \). Consider a structure \( \mathcal{M} \) with \( \mathcal{M} \models \Gamma_1 \cup \Gamma_2 \). By induction hypothesis, \( \mathcal{M} \models \varphi(t_1) \) and \( \mathcal{M} \models t_1 = t_2 \). Therefore, \( \text{Val}^\mathcal{M}(t_1) = \text{Val}^\mathcal{M}(t_2) \). Let \( s \) be any variable assignment, and \( s' \) be the \( x \)-variant given by \( s'(x) = \text{Val}^\mathcal{M}(t_1) = \text{Val}^\mathcal{M}(t_2) \). By ??, \( \mathcal{M}, s \models \varphi(t_1) \) iff \( \mathcal{M}, s' \models \varphi(x) \) iff \( \mathcal{M}, s \models \varphi(t_2) \). Since \( \mathcal{M} \models \varphi(t_1) \), we have \( \mathcal{M} \models \varphi(t_2) \). \( \square \)

Photo Credits

Bibliography