

## ntd.1 Soundness with Identity predicate

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**Proposition ntd.1.** *Natural deduction with rules for = is sound.*

*Proof.* Any formula of the form  $t = t$  is valid, since for every structure  $\mathfrak{M}$ ,  $\mathfrak{M} \models t = t$ . (Note that we assume the term  $t$  to be ground, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a derivation is =Elim, i.e., the derivation has the following form:

$$\frac{\begin{array}{c} \Gamma_1 \\ \vdots \\ \delta_1 \\ \vdots \\ t_1 = t_2 \end{array} \quad \begin{array}{c} \Gamma_2 \\ \vdots \\ \delta_2 \\ \vdots \\ \varphi(t_1) \end{array}}{\varphi(t_2)} =\text{Elim}$$

The premises  $t_1 = t_2$  and  $\varphi(t_1)$  are derived from undischarged assumptions  $\Gamma_1$  and  $\Gamma_2$ , respectively. We want to show that  $\varphi(t_2)$  follows from  $\Gamma_1 \cup \Gamma_2$ . Consider a structure  $\mathfrak{M}$  with  $\mathfrak{M} \models \Gamma_1 \cup \Gamma_2$ . By induction hypothesis,  $\mathfrak{M} \models \varphi(t_1)$  and  $\mathfrak{M} \models t_1 = t_2$ . Therefore,  $\text{Val}^{\mathfrak{M}}(t_1) = \text{Val}^{\mathfrak{M}}(t_2)$ . Let  $s$  be any variable assignment, and  $s'$  be the  $x$ -variant given by  $s'(x) = \text{Val}^{\mathfrak{M}}(t_1) = \text{Val}^{\mathfrak{M}}(t_2)$ . By ??,  $\mathfrak{M}, s \models \varphi(t_1)$  iff  $\mathfrak{M}, s' \models \varphi(x)$  iff  $\mathfrak{M}, s \models \varphi(t_2)$ . Since  $\mathfrak{M} \models \varphi(t_1)$ , we have  $\mathfrak{M} \models \varphi(t_2)$ .  $\square$

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**Bibliography**