ntd.1 Quantifier Rules

fol:ntd:qrl:sec

Rules for $\forall$

$$\frac{\varphi(a)}{\forall x\, \varphi(x)}\ \forall\text{Intro} \qquad\qquad \frac{\forall x\, \varphi(x)}{\varphi(t)}\ \forall\text{Elim}$$

In the rules for $\forall$, t is a closed term (a term that does not contain any variables), and a is a constant symbol which does not occur in the conclusion $\forall x\, \varphi(x)$, or in any assumption which is undischarged in the derivation ending with the premise $\varphi(a)$. We call a the *eigenvariable* of the $\forall$Intro inference.[1]

Rules for $\exists$

$$\frac{\varphi(t)}{\exists x\, \varphi(x)}\ \exists\text{Intro} \qquad\qquad {}_{n}\,\frac{\exists x\, \varphi(x) \qquad \begin{array}{c}[\varphi(a)]^n \\ \vdots \\ \chi\end{array}}{\chi}\ \exists\text{Elim}$$

Again, t is a closed term, and a is a constant which does not occur in the premise $\exists x\, \varphi(x)$, in the conclusion χ, or any assumption which is undischarged in the derivations ending with the two premises (other than the assumptions $\varphi(a)$). We call a the *eigenvariable* of the $\exists$Elim inference.

The condition that an eigenvariable neither occur in the premises nor in any assumption that is undischarged in the derivations leading to the premises for the $\forall$Intro or $\exists$Elim inference is called the *eigenvariable condition*.

Recall the convention that when φ is a formula with the variable x free, we indicate this by writing $\varphi(x)$. In the same context, $\varphi(t)$ then is short for $\varphi[t/x]$. So we could also write the $\exists$Intro rule as: explanation

$$\frac{\varphi[t/x]}{\exists x\, \varphi}\ \exists\text{Intro}$$

Note that t may already occur in φ, e.g., φ might be $P(t, x)$. Thus, inferring $\exists x\, P(t, x)$ from $P(t, t)$ is a correct application of $\exists$Intro—you may "replace" one or more, and not necessarily all, occurrences of t in the premise by the bound variable x. However, the eigenvariable conditions in $\forall$Intro and $\exists$Elim require that the constant symbol a does not occur in φ. So, you cannot correctly infer $\forall x\, P(a, x)$ from $P(a, a)$ using $\forall$Intro.

[1] We use the term "eigenvariable" even though a in the above rule is a constant. This has historical reasons.

explanation

In ∃Intro and ∀Elim there are no restrictions, and the term t can be anything, so we do not have to worry about any conditions. On the other hand, in the ∃Elim and ∀Intro rules, the eigenvariable condition requires that the constant symbol a does not occur anywhere in the conclusion or in an undischarged assumption. The condition is necessary to ensure that the system is sound, i.e., only derives sentences from undischarged assumptions from which they follow. Without this condition, the following would be allowed:

$$\dfrac{\exists x\, \varphi(x) \qquad \dfrac{[\varphi(a)]^1}{\forall x\, \varphi(x)}\ {*}\forall\text{Intro}}{\forall x\, \varphi(x)}\ \exists\text{Elim}$$

However, $\exists x\, \varphi(x) \nvDash \forall x\, \varphi(x)$.

As the elimination rules for quantifiers only allow substituting closed terms for variables, it follows that any formula that can be derived from a set of sentences is itself a sentence.

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Bibliography