

ntd.1 Examples of Derivations

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sec

Example ntd.1. Let's give a **derivation** of the **sentence** $(\varphi \wedge \psi) \rightarrow \varphi$.

We begin by writing the desired conclusion at the bottom of the **derivation**.

$$\frac{}{(\varphi \wedge \psi) \rightarrow \varphi}$$

Next, we need to figure out what kind of inference could result in a **sentence** of this form. The **main operator** of the conclusion is \rightarrow , so we'll try to arrive at the conclusion using the \rightarrow Intro rule. It is best to write down the assumptions involved and label the inference rules as you progress, so it is easy to see whether all assumptions have been **discharged** at the end of the proof.

$$1 \frac{\begin{array}{c} [\varphi \wedge \psi]^1 \\ \vdots \\ \vdots \\ \varphi \end{array}}{(\varphi \wedge \psi) \rightarrow \varphi} \rightarrow\text{Intro}$$

We now need to fill in the steps from the assumption $\varphi \wedge \psi$ to φ . Since we only have one connective to deal with, \wedge , we must use the \wedge elim rule. This gives us the following proof:

$$1 \frac{\frac{[\varphi \wedge \psi]^1}{\varphi} \wedge\text{Elim}}{(\varphi \wedge \psi) \rightarrow \varphi} \rightarrow\text{Intro}$$

We now have a correct **derivation** of $(\varphi \wedge \psi) \rightarrow \varphi$.

Example ntd.2. Now let's give a **derivation** of $(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$.

We begin by writing the desired conclusion at the bottom of the derivation.

$$\frac{}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)}$$

To find a logical rule that could give us this conclusion, we look at the logical connectives in the conclusion: \neg , \vee , and \rightarrow . We only care at the moment about the first occurrence of \rightarrow because it is the **main operator** of the **sentence** in the end-sequent, while \neg , \vee and the second occurrence of \rightarrow are inside the scope of another connective, so we will take care of those later. We therefore start with the \rightarrow Intro rule. A correct application must look like this:

$$1 \frac{\begin{array}{c} [\neg\varphi \vee \psi]^1 \\ \vdots \\ \vdots \\ \varphi \rightarrow \psi \end{array}}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}$$

This leaves us with two possibilities to continue. Either we can keep working from the bottom up and look for another application of the \rightarrow Intro rule, or we can work from the top down and apply a \vee Elim rule. Let us apply the latter. We will use the assumption $\neg\varphi \vee \psi$ as the leftmost premise of \vee Elim. For a valid application of \vee Elim, the other two premises must be identical to the conclusion $\varphi \rightarrow \psi$, but each may be derived in turn from another assumption, namely one of the two disjuncts of $\neg\varphi \vee \psi$. So our **derivation** will look like this:

$$\begin{array}{c}
 \begin{array}{c} [\neg\varphi]^2 \\ \vdots \\ \varphi \rightarrow \psi \end{array} \quad \begin{array}{c} [\psi]^2 \\ \vdots \\ \varphi \rightarrow \psi \end{array} \\
 \hline
 2 \frac{[\neg\varphi \vee \psi]^1 \quad \varphi \rightarrow \psi \quad \varphi \rightarrow \psi}{\varphi \rightarrow \psi} \vee\text{Elim} \\
 \hline
 1 \frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}
 \end{array}$$

In each of the two branches on the right, we want to **derive** $\varphi \rightarrow \psi$, which is best done using \rightarrow Intro.

$$\begin{array}{c}
 \begin{array}{c} [\neg\varphi]^2, [\varphi]^3 \\ \vdots \\ \psi \end{array} \quad \begin{array}{c} [\psi]^2, [\varphi]^4 \\ \vdots \\ \psi \end{array} \\
 \hline
 3 \frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \quad 4 \frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \\
 \hline
 2 \frac{[\neg\varphi \vee \psi]^1 \quad \varphi \rightarrow \psi \quad \varphi \rightarrow \psi}{\varphi \rightarrow \psi} \vee\text{Elim} \\
 \hline
 1 \frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}
 \end{array}$$

For the two missing parts of the **derivation**, we need **derivations** of ψ from $\neg\varphi$ and φ in the middle, and from φ and ψ on the left. Let's take the former first. $\neg\varphi$ and φ are the two premises of \neg Elim:

$$\begin{array}{c}
 \frac{[\neg\varphi]^2 \quad [\varphi]^3}{\perp} \neg\text{Elim} \\
 \vdots \\
 \psi
 \end{array}$$

By using \perp_I , we can obtain ψ as a conclusion and complete the branch.

$$\begin{array}{c}
 \begin{array}{c} [\neg\varphi]^2 \quad [\varphi]^3 \\ \vdots \\ \perp \end{array} \quad \begin{array}{c} [\psi]^2, [\varphi]^4 \\ \vdots \\ \psi \end{array} \\
 \hline
 \frac{\perp}{\psi} \perp_I \quad \frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \quad 4 \frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \\
 \hline
 3 \frac{\frac{\perp}{\psi} \perp_I \quad \psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \quad \varphi \rightarrow \psi \\
 \hline
 2 \frac{[\neg\varphi \vee \psi]^1 \quad \varphi \rightarrow \psi \quad \varphi \rightarrow \psi}{\varphi \rightarrow \psi} \vee\text{Elim} \\
 \hline
 1 \frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}
 \end{array}$$

Let's now look at the rightmost branch. Here it's important to realize that the definition of **derivation** allows assumptions to be discharged but does not require them to be. In other words, if we can derive ψ from one of the assumptions φ and ψ without using the other, that's ok. And to **derive** ψ from ψ is trivial: ψ by itself is such a **derivation**, and no inferences are needed. So we can simply delete the assumption φ .

$$\begin{array}{c}
 \frac{[\neg\varphi]^2 \quad [\varphi]^3}{\perp} \neg\text{Elim} \\
 \frac{\frac{\perp}{\psi} \perp_I}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \quad \frac{[\psi]^2}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \\
 \frac{[\neg\varphi \vee \psi]^1}{\varphi \rightarrow \psi} \vee\text{Elim} \\
 \frac{1 \quad \varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}
 \end{array}$$

Note that in the finished **derivation**, the rightmost \rightarrow Intro inference does not actually discharge any assumptions.

Example ntd.3. So far we have not needed the \perp_C rule. It is special in that it allows us to discharge an assumption that isn't a sub-formula of the conclusion of the rule. It is closely related to the \perp_I rule. In fact, the \perp_I rule is a special case of the \perp_C rule—there is a logic called “intuitionistic logic” in which only \perp_I is allowed. The \perp_C rule is a last resort when nothing else works. For instance, suppose we want to **derive** $\varphi \vee \neg\varphi$. Our usual strategy would be to attempt to **derive** $\varphi \vee \neg\varphi$ using \vee Intro. But this would require us to **derive** either φ or $\neg\varphi$ from no assumptions, and this can't be done. \perp_C to the rescue!

$$\frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \vdots}{\perp} \perp_C$$

Now we're looking for a **derivation** of \perp from $\neg(\varphi \vee \neg\varphi)$. Since \perp is the conclusion of \neg Elim we might try that:

$$\frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \vdots \quad \neg\varphi \quad [\neg(\varphi \vee \neg\varphi)]^1 \quad \vdots \quad \varphi}{\perp} \neg\text{Elim} \\
 \frac{1 \quad \perp}{\varphi \vee \neg\varphi} \perp_C$$

Our strategy for finding a **derivation** of $\neg\varphi$ calls for an application of \neg Intro:

$$\begin{array}{c}
[\neg(\varphi \vee \neg\varphi)]^1, [\varphi]^2 \\
\vdots \\
2 \frac{\perp}{\neg\varphi} \neg\text{Intro} \qquad [\neg(\varphi \vee \neg\varphi)]^1 \\
\vdots \\
\varphi \neg\text{Elim} \\
\hline
1 \frac{\perp}{\varphi \vee \neg\varphi} \perp_C
\end{array}$$

Here, we can get \perp easily by applying $\neg\text{Elim}$ to the assumption $\neg(\varphi \vee \neg\varphi)$ and $\varphi \vee \neg\varphi$ which follows from our new assumption φ by $\vee\text{Intro}$:

$$\begin{array}{c}
[\neg(\varphi \vee \neg\varphi)]^1 \qquad \frac{[\varphi]^2}{\varphi \vee \neg\varphi} \vee\text{Intro} \qquad [\neg(\varphi \vee \neg\varphi)]^1 \\
\hline
\frac{[\neg(\varphi \vee \neg\varphi)]^1}{2 \frac{\perp}{\neg\varphi} \neg\text{Intro}} \neg\text{Elim} \qquad \vdots \\
\varphi \neg\text{Elim} \\
\hline
1 \frac{\perp}{\varphi \vee \neg\varphi} \perp_C
\end{array}$$

On the right side we use the same strategy, except we get φ by \perp_C :

$$\begin{array}{c}
[\neg(\varphi \vee \neg\varphi)]^1 \qquad \frac{[\varphi]^2}{\varphi \vee \neg\varphi} \vee\text{Intro} \qquad [\neg(\varphi \vee \neg\varphi)]^1 \qquad \frac{[\neg\varphi]^3}{\varphi \vee \neg\varphi} \vee\text{Intro} \\
\hline
2 \frac{\perp}{\neg\varphi} \neg\text{Intro} \qquad \frac{[\neg(\varphi \vee \neg\varphi)]^1}{3 \frac{\perp}{\varphi} \perp_C} \neg\text{Elim} \\
\hline
1 \frac{\perp}{\varphi \vee \neg\varphi} \perp_C
\end{array}$$

Problem ntd.1. Give **derivations** that show the following:

1. $\varphi \wedge (\psi \wedge \chi) \vdash (\varphi \wedge \psi) \wedge \chi$.
2. $\varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi$.
3. $\varphi \rightarrow (\psi \rightarrow \chi) \vdash \psi \rightarrow (\varphi \rightarrow \chi)$.
4. $\varphi \vdash \neg\neg\varphi$.

Problem ntd.2. Give **derivations** that show the following:

1. $(\varphi \vee \psi) \rightarrow \chi \vdash \varphi \rightarrow \chi$.
2. $(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \vdash (\varphi \vee \psi) \rightarrow \chi$.
3. $\vdash \neg(\varphi \wedge \neg\varphi)$.
4. $\psi \rightarrow \varphi \vdash \neg\varphi \rightarrow \neg\psi$.
5. $\vdash (\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi$.
6. $\vdash \neg(\varphi \rightarrow \psi) \rightarrow \neg\psi$.

7. $\varphi \rightarrow \chi \vdash \neg(\varphi \wedge \neg\chi)$.
8. $\varphi \wedge \neg\chi \vdash \neg(\varphi \rightarrow \chi)$.
9. $\varphi \vee \psi, \neg\psi \vdash \varphi$.
10. $\neg\varphi \vee \neg\psi \vdash \neg(\varphi \wedge \psi)$.
11. $\vdash (\neg\varphi \wedge \neg\psi) \rightarrow \neg(\varphi \vee \psi)$.
12. $\vdash \neg(\varphi \vee \psi) \rightarrow (\neg\varphi \wedge \neg\psi)$.

Problem ntd.3. Give **derivations** that show the following:

1. $\neg(\varphi \rightarrow \psi) \vdash \varphi$.
2. $\neg(\varphi \wedge \psi) \vdash \neg\varphi \vee \neg\psi$.
3. $\varphi \rightarrow \psi \vdash \neg\varphi \vee \psi$.
4. $\vdash \neg\neg\varphi \rightarrow \varphi$.
5. $\varphi \rightarrow \psi, \neg\varphi \rightarrow \psi \vdash \psi$.
6. $(\varphi \wedge \psi) \rightarrow \chi \vdash (\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)$.
7. $(\varphi \rightarrow \psi) \rightarrow \varphi \vdash \varphi$.
8. $\vdash (\varphi \rightarrow \psi) \vee (\psi \rightarrow \chi)$.

(These all require the \perp_C rule.)

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Bibliography