

## ntd.1 Examples of Derivations

fol:ntd:pro:  
sec

**Example ntd.1.** Let's give a **derivation** of the **sentence**  $(\varphi \wedge \psi) \rightarrow \varphi$ .

We begin by writing the desired conclusion at the bottom of the **derivation**.

$$\frac{}{(\varphi \wedge \psi) \rightarrow \varphi}$$

Next, we need to figure out what kind of inference could result in a **sentence** of this form. The **main operator** of the conclusion is  $\rightarrow$ , so we'll try to arrive at the conclusion using the  $\rightarrow$ Intro rule. It is best to write down the assumptions involved and label the inference rules as you progress, so it is easy to see whether all assumptions have been **discharged** at the end of the proof.

$$1 \frac{\begin{array}{c} [\varphi \wedge \psi]^1 \\ \vdots \\ \varphi \end{array}}{(\varphi \wedge \psi) \rightarrow \varphi} \rightarrow\text{Intro}$$

We now need to fill in the steps from the assumption  $\varphi \wedge \psi$  to  $\varphi$ . Since we only have one connective to deal with,  $\wedge$ , we must use the  $\wedge$  elim rule. This gives us the following proof:

$$1 \frac{\frac{[\varphi \wedge \psi]^1}{\varphi} \wedge\text{Elim}}{(\varphi \wedge \psi) \rightarrow \varphi} \rightarrow\text{Intro}$$

We now have a correct **derivation** of  $(\varphi \wedge \psi) \rightarrow \varphi$ .

**Example ntd.2.** Now let's give a **derivation** of  $(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$ .

We begin by writing the desired conclusion at the bottom of the derivation.

$$\frac{}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)}$$

To find a logical rule that could give us this conclusion, we look at the logical connectives in the conclusion:  $\neg$ ,  $\vee$ , and  $\rightarrow$ . We only care at the moment about the first occurrence of  $\rightarrow$  because it is the **main operator** of the **sentence** in the end-sequent, while  $\neg$ ,  $\vee$  and the second occurrence of  $\rightarrow$  are inside the scope of another connective, so we will take care of those later. We therefore start with the  $\rightarrow$ Intro rule. A correct application must look as follows:

$$1 \frac{\begin{array}{c} [\neg\varphi \vee \psi]^1 \\ \vdots \\ \varphi \rightarrow \psi \end{array}}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}$$

This leaves us with two possibilities to continue. Either we can keep working from the bottom up and look for another application of the  $\rightarrow$ Intro rule, or we can work from the top down and apply a  $\vee$ Elim rule. Let us apply the latter. We will use the assumption  $\neg\varphi \vee \psi$  as the leftmost premise of  $\vee$ Elim. For a valid application of  $\vee$ Elim, the other two premises must be identical to the conclusion  $\varphi \rightarrow \psi$ , but each may be derived in turn from another assumption, namely the two disjuncts of  $\neg\varphi \vee \psi$ . So our **derivation** will look like this:

$$\begin{array}{c}
 \begin{array}{ccc}
 & [\neg\varphi]^2 & [\psi]^2 \\
 & \vdots & \vdots \\
 & \varphi \rightarrow \psi & \varphi \rightarrow \psi \\
 \hline
 2 \frac{[\neg\varphi \vee \psi]^1 \quad \varphi \rightarrow \psi \quad \varphi \rightarrow \psi}{\varphi \rightarrow \psi} \vee\text{Elim} \\
 \hline
 1 \frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}
 \end{array}
 \end{array}$$

In each of the two branches on the right, we want to **derive**  $\varphi \rightarrow \psi$ , which is best done using  $\rightarrow$ Intro.

$$\begin{array}{c}
 \begin{array}{ccc}
 & [\neg\varphi]^2, [\varphi]^3 & [\psi]^2, [\varphi]^4 \\
 & \vdots & \vdots \\
 & \psi & \psi \\
 \hline
 3 \frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro} & & 4 \frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \\
 \hline
 2 \frac{[\neg\varphi \vee \psi]^1 \quad \varphi \rightarrow \psi \quad \varphi \rightarrow \psi}{\varphi \rightarrow \psi} \vee\text{Elim} \\
 \hline
 1 \frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}
 \end{array}
 \end{array}$$

For the two missing parts of the **derivation**, we need **derivations** of  $\psi$  from  $\neg\varphi$  and  $\varphi$  in the middle, and from  $\varphi$  and  $\psi$  on the left. Let's take the former first.  $\neg\varphi$  and  $\varphi$  are the two premises of  $\neg$ Elim:

$$\begin{array}{c}
 \frac{[\neg\varphi]^2 \quad [\varphi]^3}{\perp} \neg\text{Elim} \\
 \vdots \\
 \psi
 \end{array}$$

By using  $\perp_I$ , we can obtain  $\psi$  as a conclusion and complete the branch.

$$\begin{array}{c}
 \begin{array}{ccc}
 & [\neg\varphi]^2 \quad [\varphi]^3 & [\psi]^2, [\varphi]^4 \\
 & \vdots & \vdots \\
 & \perp & \psi \\
 & \perp_I & \\
 \hline
 3 \frac{\perp}{\psi} \perp_I & & 4 \frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \\
 \hline
 2 \frac{[\neg\varphi \vee \psi]^1 \quad \varphi \rightarrow \psi \quad \varphi \rightarrow \psi}{\varphi \rightarrow \psi} \vee\text{Elim} \\
 \hline
 1 \frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}
 \end{array}
 \end{array}$$

Let's now look at the rightmost branch. Here it's important to realize that the definition of **derivation** allows assumptions to be discharged but does not require them to be. In other words, if we can derive  $\psi$  from one of the assumptions  $\varphi$  and  $\psi$  without using the other, that's ok. And to **derive**  $\psi$  from  $\psi$  is trivial:  $\psi$  by itself is such a **derivation**, and no inferences are needed. So we can simply delete the assumption  $\varphi$ .

$$\begin{array}{c}
 \frac{[\neg\varphi]^2 \quad [\varphi]^3}{\perp} \neg\text{Elim} \\
 \frac{\frac{\perp}{\psi} \perp_I}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \quad \frac{[\psi]^2}{\varphi \rightarrow \psi} \rightarrow\text{Intro}}{\frac{[\neg\varphi \vee \psi]^1}{\varphi \rightarrow \psi} \vee\text{Elim}} \\
 \frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}
 \end{array}$$

Note that in the finished **derivation**, the rightmost  $\rightarrow\text{Intro}$  inference does not actually discharge any assumptions.

**Example ntd.3.** So far we have not needed the  $\perp_C$  rule. It is special in that it allows us to discharge an assumption that isn't a sub-formula of the conclusion of the rule. It is closely related to the  $\perp_I$  rule. In fact, the  $\perp_I$  rule is a special case of the  $\perp_C$  rule—there is a logic called “intuitionistic logic” in which only  $\perp_I$  is allowed. The  $\perp_C$  rule is a last resort when nothing else works. For instance, suppose we want to **derive**  $\varphi \vee \neg\varphi$ . Our usual strategy would be to attempt to **derive**  $\varphi \vee \neg\varphi$  using  $\vee\text{Intro}$ . But this would require us to **derive** either  $\varphi$  or  $\neg\varphi$  from no assumptions, and this can't be done.  $\perp_C$  to the rescue!

$$\begin{array}{c}
 [\neg(\varphi \vee \neg\varphi)]^1 \\
 \vdots \\
 \vdots \\
 \frac{\perp}{\varphi \vee \neg\varphi} \perp_C
 \end{array}$$

Now we're looking for a **derivation** of  $\perp$  from  $\neg(\varphi \vee \neg\varphi)$ . Since  $\perp$  is the conclusion of  $\neg\text{Elim}$  we might try that:

$$\begin{array}{c}
 \frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad [\neg(\varphi \vee \neg\varphi)]^1}{\frac{\neg\varphi \quad \varphi}{\perp} \neg\text{Elim}} \perp_C \\
 \frac{\perp}{\varphi \vee \neg\varphi} \perp_C
 \end{array}$$

Our strategy for finding a **derivation** of  $\neg\varphi$  calls for an application of  $\neg\text{Intro}$ :

$$\begin{array}{c}
[\neg(\varphi \vee \neg\varphi)]^1, [\varphi]^2 \\
\vdots \\
2 \frac{\perp}{\neg\varphi} \neg\text{Intro} \\
\hline
1 \frac{\perp}{\varphi \vee \neg\varphi} \perp_C \\
\hline
\begin{array}{c}
[\neg(\varphi \vee \neg\varphi)]^1 \\
\vdots \\
\varphi \neg\text{Elim}
\end{array}
\end{array}$$

Here, we can get  $\perp$  easily by applying  $\neg\text{Elim}$  to the assumption  $\neg(\varphi \vee \neg\varphi)$  and  $\varphi \vee \neg\varphi$  which follows from our new assumption  $\varphi$  by  $\vee\text{Intro}$ :

$$\begin{array}{c}
\frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \frac{[\varphi]^2}{\varphi \vee \neg\varphi} \vee\text{Intro}}{\frac{\perp}{\neg\varphi} \neg\text{Intro}} \neg\text{Elim} \\
\hline
2 \frac{\perp}{\neg\varphi} \neg\text{Intro} \\
\hline
1 \frac{\perp}{\varphi \vee \neg\varphi} \perp_C \\
\hline
\begin{array}{c}
[\neg(\varphi \vee \neg\varphi)]^1 \\
\vdots \\
\varphi \neg\text{Elim}
\end{array}
\end{array}$$

On the right side we use the same strategy, except we get  $\varphi$  by  $\perp_C$ :

$$\begin{array}{c}
\frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \frac{[\varphi]^2}{\varphi \vee \neg\varphi} \vee\text{Intro}}{\frac{\perp}{\neg\varphi} \neg\text{Intro}} \neg\text{Elim} \quad \frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \frac{[\neg\varphi]^3}{\varphi \vee \neg\varphi} \vee\text{Intro}}{\frac{\perp}{\varphi} \perp_C} \neg\text{Elim} \\
\hline
2 \frac{\perp}{\neg\varphi} \neg\text{Intro} \quad 3 \frac{\perp}{\varphi} \perp_C \\
\hline
1 \frac{\perp}{\varphi \vee \neg\varphi} \perp_C \\
\hline
\end{array}$$

**Problem ntd.1.** Give **derivations** of the following:

1.  $\neg(\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \neg\psi)$
2.  $(\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)$  from the assumption  $(\varphi \wedge \psi) \rightarrow \chi$

## Photo Credits

## Bibliography