Derivability and the Quantifiers

Theorem ntd.1. If $c$ is a constant not occurring in $\Gamma$ or $\varphi(x)$ and $\Gamma \vdash \varphi(c)$, then $\Gamma \vdash \forall x \varphi(x)$.

Proof. Let $\delta$ be a derivation of $\varphi(c)$ from $\Gamma$. By adding a $\forall$Intro inference, we obtain a proof of $\forall x \varphi(x)$. Since $c$ does not occur in $\Gamma$ or $\varphi(x)$, the eigenvariable condition is satisfied.

Proposition ntd.2.

1. $\varphi(t) \vdash \exists x \varphi(x)$.
2. $\forall x \varphi(x) \vdash \varphi(t)$.

Proof. 1. The following is a derivation of $\exists x \varphi(x)$ from $\varphi(t)$:

\[
\frac{\varphi(t)}{\exists x \varphi(x)} \exists \text{Intro}
\]

2. The following is a derivation of $\varphi(t)$ from $\forall x \varphi(x)$:

\[
\frac{\forall x \varphi(x)}{\varphi(t)} \forall \text{Elim}
\]

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Bibliography