

## ntd.1 Derivability and the Propositional Connectives

fol:ntd:ppr:sec We establish that the **derivability** relation  $\vdash$  of natural deduction is strong explanation enough to establish some basic facts involving the propositional connectives, such as that  $\varphi \wedge \psi \vdash \varphi$  and  $\varphi, \varphi \rightarrow \psi \vdash \psi$  (modus ponens). These facts are needed for the proof of the completeness theorem.

### Proposition ntd.1.

fol:ntd:ppr:prop:provability-land  
fol:ntd:ppr:prop:provability-land-left  
fol:ntd:ppr:prop:provability-land-right

1. Both  $\varphi \wedge \psi \vdash \varphi$  and  $\varphi \wedge \psi \vdash \psi$
2.  $\varphi, \psi \vdash \varphi \wedge \psi$ .

*Proof.* 1. We can **derive** both

$$\frac{\varphi \wedge \psi}{\varphi} \wedge\text{Elim} \qquad \frac{\varphi \wedge \psi}{\psi} \wedge\text{Elim}$$

2. We can **derive**:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge\text{Intro} \qquad \square$$

### Proposition ntd.2.

fol:ntd:ppr:prop:provability-lor

1.  $\varphi \vee \psi, \neg\varphi, \neg\psi$  is inconsistent.
2. Both  $\varphi \vdash \varphi \vee \psi$  and  $\psi \vdash \varphi \vee \psi$ .

*Proof.* 1. Consider the following **derivation**:

$$\frac{1 \quad \varphi \vee \psi \quad \frac{\neg\varphi \quad [\varphi]^1}{\perp} \neg\text{Elim} \quad \frac{\neg\psi \quad [\psi]^1}{\perp} \neg\text{Elim}}{\perp} \vee\text{Elim}$$

This is a **derivation** of  $\perp$  from **undischarged** assumptions  $\varphi \vee \psi$ ,  $\neg\varphi$ , and  $\neg\psi$ .

2. We can **derive** both

$$\frac{\varphi}{\varphi \vee \psi} \vee\text{Intro} \qquad \frac{\psi}{\varphi \vee \psi} \vee\text{Intro} \qquad \square$$

### Proposition ntd.3.

fol:ntd:ppr:prop:provability-lif  
fol:ntd:ppr:prop:provability-lif-left  
fol:ntd:ppr:prop:provability-lif-right

1.  $\varphi, \varphi \rightarrow \psi \vdash \psi$ .
2. Both  $\neg\varphi \vdash \varphi \rightarrow \psi$  and  $\psi \vdash \varphi \rightarrow \psi$ .

*Proof.* 1. We can **derive**:

$$\frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \rightarrow\text{Elim}$$

2. This is shown by the following two **derivations**:

$$\frac{\frac{\frac{\neg\varphi \quad [\varphi]^1}{\perp} \neg\text{Elim}}{\psi} \perp_I}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \quad \frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro}$$

Note that  $\rightarrow\text{Intro}$  may, but does not have to, **discharge** the assumption  $\varphi$ .  
 $\square$

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## Bibliography