Derivability and the Propositional Connectives

We establish that the derivability relation $\vdash$ of natural deduction is strong enough to establish some basic facts involving the propositional connectives, such as that $\phi \land \psi \vdash \phi$ and $\phi, \phi \rightarrow \psi \vdash \psi$ (modus ponens). These facts are needed for the proof of the completeness theorem.

**Proposition ntd.1.**

1. Both $\phi \land \psi \vdash \phi$ and $\phi \land \psi \vdash \psi$
2. $\phi, \psi \vdash \phi \land \psi$.

**Proof.** 1. We can derive both

$$
\frac{\phi \land \psi}{\phi} \ \land\text{Elim} \quad \frac{\phi \land \psi}{\psi} \ \land\text{Elim}
$$

2. We can derive:

$$
\frac{\phi \quad \psi}{\phi \land \psi} \ \land\text{Intro}
$$

**Proposition ntd.2.**

1. $\phi \lor \psi, \neg \phi, \neg \psi$ is inconsistent.
2. Both $\phi \vdash \phi \lor \psi$ and $\psi \vdash \phi \lor \psi$.

**Proof.** 1. Consider the following derivation:

$$
\frac{\phi \lor \psi \quad \neg \phi \quad \neg \psi}{\bot} \ \land\text{Elim} \quad \frac{\neg \psi}{\bot} \ \lor\text{Elim}
$$

This is a derivation of $\bot$ from undischarged assumptions $\phi \lor \psi, \neg \phi$, and $\neg \psi$.

2. We can derive both

$$
\frac{\phi \lor \psi}{\phi \lor \psi} \ \lor\text{Intro} \quad \frac{\psi}{\phi \lor \psi} \ \lor\text{Intro}
$$

**Proposition ntd.3.**

1. $\phi, \phi \rightarrow \psi \vdash \psi$.
2. Both $\neg \phi \vdash \phi \rightarrow \psi$ and $\psi \vdash \phi \rightarrow \psi$.
Proof. 1. We can derive:

\[
\frac{\varphi \rightarrow \psi}{\varphi} \quad \text{ Elim}
\]

2. This is shown by the following two derivations:

\[
\begin{align*}
\neg \varphi & \quad [\varphi]^1 \quad \text{ Elim} \\
\hline
\bot & \quad \bot_I \\
1 \varphi \rightarrow \psi & \quad \text{ Intro} \\
\hline
\psi & \quad \psi \rightarrow \psi \quad \text{ Intro}
\end{align*}
\]

Note that \( \rightarrow \text{Intro} \) may, but does not have to, discharge the assumption \( \varphi \).

\( \square \)

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Bibliography