## ntd.1 Derivability and Consistency

fol:ntd:prv: sec

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

*fol:ntd:prv: prop:provability-contr*

**Proposition ntd.1.** *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.*

*Proof.* Let the derivation of $\varphi$ from $\Gamma$ be $\delta_1$ and the derivation of $\bot$ from $\Gamma \cup \{\varphi\}$ be $\delta_2$. We can then derive:

$$\dfrac{{\scriptstyle 1}\,\dfrac{\begin{matrix}\Gamma, [\varphi]^1\\ \vdots\ \delta_2\\ \bot\end{matrix}}{\neg\varphi}\ \neg\mathrm{Intro} \qquad \begin{matrix}\Gamma\\ \vdots\ \delta_1\\ \varphi\end{matrix}}{\bot}\ \neg\mathrm{Elim}$$

In the new derivation, the assumption $\varphi$ is discharged, so it is a derivation from $\Gamma$. □

*fol:ntd:prv: prop:prov-incons*

**Proposition ntd.2.** *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

*Proof.* First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation $\delta_0$ of $\varphi$ from undischarged assumptions $\Gamma$. We obtain a derivation of $\bot$ from $\Gamma \cup \{\neg\varphi\}$ as follows:

$$\dfrac{\neg\varphi \qquad \begin{matrix}\Gamma\\ \vdots\ \delta_0\\ \varphi\end{matrix}}{\bot}\ \neg\mathrm{Elim}$$

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, and let $\delta_1$ be the corresponding derivation of $\bot$ from undischarged assumptions in $\Gamma \cup \{\neg\varphi\}$. We obtain a derivation of $\varphi$ from $\Gamma$ alone by using $\bot_C$:

$${\scriptstyle 1}\,\dfrac{\begin{matrix}\Gamma, [\neg\varphi]^1\\ \vdots\ \delta_1\\ \bot\end{matrix}}{\varphi}\ \bot_C$$

□

**Problem ntd.1.** Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

*fol:ntd:prv: prop:explicit-inc*

**Proposition ntd.3.** *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then $\Gamma$ is inconsistent.*

*Proof.* Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a derivation $\delta$ of $\varphi$ from $\Gamma$. Consider this simple application of the $\neg$Elim rule:

$$\dfrac{\neg\varphi \qquad \begin{matrix}\Gamma\\ \vdots\,\delta \\ \varphi\end{matrix}}{\bot}\,\neg\mathrm{Elim}$$

Since $\neg\varphi \in \Gamma$, all undischarged assumptions are in $\Gamma$, this shows that $\Gamma \vdash \bot$. □

**Proposition ntd.4.** *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.* fol:ntd:prv: prop:provability-exhaustive

*Proof.* There are derivations $\delta_1$ and $\delta_2$ of $\bot$ from $\Gamma \cup \{\varphi\}$ and $\bot$ from $\Gamma \cup \{\neg\varphi\}$, respectively. We can then derive

$$\dfrac{{}^2\,\dfrac{\begin{matrix}\Gamma,[\neg\varphi]^2\\ \vdots\,\delta_2\\ \bot\end{matrix}}{\neg\neg\varphi}\,\neg\mathrm{Intro} \qquad {}^1\,\dfrac{\begin{matrix}\Gamma,[\varphi]^1\\ \vdots\,\delta_1\\ \bot\end{matrix}}{\neg\varphi}\,\neg\mathrm{Intro}}{\bot}\,\neg\mathrm{Elim}$$

Since the assumptions $\varphi$ and $\neg\varphi$ are discharged, this is a derivation of $\bot$ from $\Gamma$ alone. Hence $\Gamma$ is inconsistent. □

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## Bibliography