

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a **derivation** δ of φ from Γ . Consider this simple application of the \neg Elim rule:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \delta \\ \vdots \\ \varphi \end{array} \quad \neg\varphi}{\perp} \neg\text{Elim}$$

Since $\neg\varphi \in \Gamma$, all **undischarged** assumptions are in Γ , this shows that $\Gamma \vdash \perp$. \square

Proposition ntd.4. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.* *fol.ntd:prv:
prop:provability-exhaustive*

Proof. There are **derivations** δ_1 and δ_2 of \perp from $\Gamma \cup \{\varphi\}$ and \perp from $\Gamma \cup \{\neg\varphi\}$, respectively. We can then **derive**

$$\frac{\begin{array}{c} \Gamma, [\neg\varphi]^2 \\ \vdots \\ \delta_2 \\ \vdots \\ \perp \end{array} \quad \begin{array}{c} \Gamma, [\varphi]^1 \\ \vdots \\ \delta_1 \\ \vdots \\ \perp \end{array}}{\frac{2 \frac{\perp}{\neg\neg\varphi} \neg\text{Intro} \quad 1 \frac{\perp}{\neg\varphi} \neg\text{Intro}}{\perp} \neg\text{Elim}}$$

Since the assumptions φ and $\neg\varphi$ are **discharged**, this is a **derivation** of \perp from Γ alone. Hence Γ is inconsistent. \square

Photo Credits

Bibliography