ntd.1 Derivability and Consistency

fol:ntd:prv:sec

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:ntd:prv:prop:provability-contr

Proposition ntd.1. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Proof. Let the derivation of φ from Γ be δ_1 and the derivation of $\bot$ from $\Gamma \cup \{\varphi\}$ be δ_2. We can then derive:

$$\dfrac{{}^1\,\dfrac{\begin{array}{c}\Gamma, [\varphi]^1\\ \vdots\ \delta_2\\ \bot\end{array}}{\neg\varphi}\ \neg\text{Intro} \qquad \begin{array}{c}\Gamma\\ \vdots\ \delta_1\\ \varphi\end{array}}{\bot}\ \neg\text{Elim}$$

In the new derivation, the assumption φ is discharged, so it is a derivation from Γ. □

fol:ntd:prv:prop:prov-incons

Proposition ntd.2. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation δ_0 of φ from undischarged assumptions Γ. We obtain a derivation of $\bot$ from $\Gamma \cup \{\neg\varphi\}$ as follows:

$$\dfrac{\neg\varphi \qquad \begin{array}{c}\Gamma\\ \vdots\ \delta_0\\ \varphi\end{array}}{\bot}\ \neg\text{Elim}$$

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, and let δ_1 be the corresponding derivation of $\bot$ from undischarged assumptions in $\Gamma \cup \{\neg\varphi\}$. We obtain a derivation of φ from Γ alone by using $\bot_C$:

$$\dfrac{\begin{array}{c}\Gamma, [\neg\varphi]^1\\ \vdots\ \delta_1\\ \bot\end{array}}{\varphi}\ \bot_C$$

□

Problem ntd.1. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:ntd:prv:prop:explicit-inc

Proposition ntd.3. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a derivation δ of φ from Γ. Consider this simple application of the $\neg$Elim rule:

$$\dfrac{\neg\varphi \qquad \begin{matrix}\Gamma\\ \vdots\,\delta \\ \varphi\end{matrix}}{\bot}\;\neg\mathrm{Elim}$$

Since $\neg\varphi \in \Gamma$, all undischarged assumptions are in Γ, this shows that $\Gamma \vdash \bot$. ☐

Proposition ntd.4. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.* *fol:ntd:prv: prop:provability-exhaustive*

Proof. There are derivations δ_1 and δ_2 of $\bot$ from $\Gamma \cup \{\varphi\}$ and $\bot$ from $\Gamma \cup \{\neg\varphi\}$, respectively. We can then derive

$$\dfrac{{}^{2}\,\dfrac{\begin{matrix}\Gamma, [\neg\varphi]^2\\ \vdots\,\delta_2\\ \bot\end{matrix}}{\neg\neg\varphi}\;\neg\mathrm{Intro} \qquad {}^{1}\,\dfrac{\begin{matrix}\Gamma, [\varphi]^1\\ \vdots\,\delta_1\\ \bot\end{matrix}}{\neg\varphi}\;\neg\mathrm{Intro}}{\bot}\;\neg\mathrm{Elim}$$

Since the assumptions φ and $\neg\varphi$ are discharged, this is a derivation of $\bot$ from Γ alone. Hence Γ is inconsistent. ☐

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Bibliography