

ntd.1 Derivability and Consistency

fol:ntd:prv:
sec

We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:ntd:prv:
prop:provability-contr

Proposition ntd.1. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Proof. Let the **derivation** of φ from Γ be δ_1 and the **derivation** of \perp from $\Gamma \cup \{\varphi\}$ be δ_2 . We can then **derive**:

$$\frac{\begin{array}{c} \Gamma, [\varphi]^1 \\ \vdots \\ \delta_2 \\ \vdots \\ \perp \\ \hline 1 \frac{\perp}{\neg\varphi} \text{-Intro} \end{array} \quad \begin{array}{c} \Gamma \\ \vdots \\ \delta_1 \\ \vdots \\ \varphi \\ \hline \text{-Elim} \end{array}}{\perp} \text{-Elim}$$

In the new **derivation**, the assumption φ is **discharged**, so it is a **derivation** from Γ . □

fol:ntd:prv:
prop:prov-incons

Proposition ntd.2. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a **derivation** δ_0 of φ from **undischarged** assumptions Γ . We obtain a **derivation** of \perp from $\Gamma \cup \{\neg\varphi\}$ as follows:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \delta_0 \\ \vdots \\ \varphi \\ \hline \neg\varphi \end{array}}{\perp} \text{-Elim}$$

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, and let δ_1 be the corresponding derivation of \perp from **undischarged** assumptions in $\Gamma \cup \{\neg\varphi\}$. We obtain a **derivation** of φ from Γ alone by using \perp_C :

$$\frac{\begin{array}{c} \Gamma, [\neg\varphi]^1 \\ \vdots \\ \delta_1 \\ \vdots \\ \perp \\ \hline \frac{\perp}{\varphi} \perp_C \end{array}}{\varphi} \perp_C$$

□

Problem ntd.1. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

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prop:explicit-inc

Proposition ntd.3. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a **derivation** δ of φ from Γ . Consider this simple application of the \neg Elim rule:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \delta \\ \vdots \\ \varphi \end{array}}{\perp} \neg\text{Elim}$$

Since $\neg\varphi \in \Gamma$, all **undischarged** assumptions are in Γ , this shows that $\Gamma \vdash \perp$. \square

Proposition ntd.4. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.* *fol.ntd:prv:
prop:provability-exhaustive*

Proof. There are **derivations** δ_1 and δ_2 of \perp from $\Gamma \cup \{\varphi\}$ and \perp from $\Gamma \cup \{\neg\varphi\}$, respectively. We can then **derive**

$$\frac{\begin{array}{c} \Gamma, [\neg\varphi]^2 \\ \vdots \\ \delta_2 \\ \vdots \\ \perp \end{array} \quad \begin{array}{c} \Gamma, [\varphi]^1 \\ \vdots \\ \delta_1 \\ \vdots \\ \perp \end{array}}{\perp} \frac{\frac{2 \quad \perp}{\neg\neg\varphi} \neg\text{Intro} \quad 1 \quad \frac{\perp}{\neg\varphi} \neg\text{Intro}}{\perp} \neg\text{Elim}$$

Since the assumptions φ and $\neg\varphi$ are **discharged**, this is a **derivation** of \perp from Γ alone. Hence Γ is inconsistent. \square

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Bibliography