

ntd.1 Proof-Theoretic Notions

fol:ntd:ptn:
sec

This section collects the definitions the provability relation and consistency for natural deduction.

Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain sentences from others. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*.

explanation

Definition ntd.1 (Theorems). A sentence φ is a *theorem* if there is a *derivation* of φ in natural deduction in which all assumptions are *discharged*. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition ntd.2 (Derivability). A sentence φ is *derivable from* a set of sentences Γ , $\Gamma \vdash \varphi$, if there is a *derivation* with conclusion φ and in which every assumption is either *discharged* or is in Γ . If φ is not *derivable* from Γ we write $\Gamma \not\vdash \varphi$.

Definition ntd.3 (Consistency). A set of sentences Γ is *inconsistent* iff $\Gamma \vdash \perp$. If Γ is not inconsistent, i.e., if $\Gamma \not\vdash \perp$, we say it is *consistent*.

Proposition ntd.4 (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

fol:ntd:ptn:
prop:reflexivity

Proof. The assumption φ by itself is a *derivation* of φ where every *undischarged* assumption (i.e., φ) is in Γ . □

Proposition ntd.5 (Monotony). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

fol:ntd:ptn:
prop:monotony

Proof. Any *derivation* of φ from Γ is also a *derivation* of φ from Δ . □

Proposition ntd.6 (Transitivity). *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

fol:ntd:ptn:
prop:transitivity

Proof. If $\Gamma \vdash \varphi$, there is a *derivation* δ_0 of φ with all *undischarged* assumptions in Γ . If $\{\varphi\} \cup \Delta \vdash \psi$, then there is a *derivation* δ_1 of ψ with all *undischarged* assumptions in $\{\varphi\} \cup \Delta$. Now consider:

$$\begin{array}{c}
 \Delta, [\varphi]^1 \\
 \vdots \\
 \delta_1 \\
 \vdots \\
 \psi \\
 \hline
 \varphi \rightarrow \psi \quad \rightarrow\text{Intro} \\
 \hline
 \psi
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma \\
 \vdots \\
 \delta_0 \\
 \vdots \\
 \varphi \\
 \hline
 \varphi \quad \rightarrow\text{Elim} \\
 \hline
 \psi
 \end{array}$$

The **undischarged** assumptions are now all among $\Gamma \cup \Delta$, so this shows $\Gamma \cup \Delta \vdash \psi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i , then $\Gamma \vdash \psi$.

Proposition ntd.7. Γ is inconsistent iff $\Gamma \vdash \varphi$ for every *sentence* φ .

*fol.ntd.ptn:
prop:incons*

Proof. Exercise. □

Problem ntd.1. Prove [Proposition ntd.7](#)

Proposition ntd.8 (Compactness).

*fol.ntd.ptn:
prop:proves-compact*

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of Γ is consistent, then Γ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a **derivation** δ of φ from Γ . Let Γ_0 be the set of **undischarged** assumptions of δ . Since any **derivation** is finite, Γ_0 can only contain finitely many **sentences**. So, δ is a **derivation** of φ from a finite $\Gamma_0 \subseteq \Gamma$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \perp$. □

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Bibliography