ntd.1 Proof-Theoretic Notions

fol:ntd:ptn:sec Just as we've defined a number of important semantic notions (validity, entailment, satisfiabilty), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain sentences from others. It was an important discovery, due to Gödel, that these notions coincide. That they do is the content of the *completeness theorem*. explanation

Definition ntd.1 (Theorems)**.** A sentence φ is a *theorem* if there is a derivation of φ in natural deduction in which all assumptions are discharged. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Definition ntd.2 (Derivability)**.** A sentence φ is *derivable from* a set of sentences Γ, $\Gamma \vdash \varphi$, if there is a derivation with conclusion φ and in which every assumption is either discharged or is in Γ. If φ is not derivable from Γ we write $\Gamma \nvdash \varphi$.

Definition ntd.3 (Consistency)**.** A set of sentences Γ is *inconsistent* iff $\Gamma \vdash \bot$. If Γ is not inconsistent, i.e., if $\Gamma \nvdash \bot$, we say it is *consistent*.

fol:ntd:ptn:prop:reflexivity **Proposition ntd.4** (Reflexivity)**.** *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. The assumption φ by itself is a derivation of φ where every undischarged assumption (i.e., φ) is in Γ. □

fol:ntd:ptn:prop:monotony **Proposition ntd.5** (Monotony)**.** *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any derivation of φ from Γ is also a derivation of φ from Δ. □

fol:ntd:ptn:prop:transitivity **Proposition ntd.6** (Transitivity)**.** *If $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$ and $\Delta \vdash \psi$, then $\Gamma \vdash \psi$.*

Proof. If $\Delta \vdash \psi$, then there is a derivation δ_0 of ψ with all undischarged assumptions in Δ. We show that $\Gamma \vdash \psi$ by induction on the number n of undischarged assumptions in δ_0.

If $n = 0$, then δ_0 has no undischarged assumptions, and so also counts as a derivation of ψ from Γ.

Otherwise, pick an undischarged assumption φ in δ_0 and let Δ_1 be the remaining undischarged assumptions. We obtain the derivation δ_1:

$$\dfrac{\begin{array}{c}\Delta_1, [\varphi]^1 \\ \vdots\ \delta_0 \\ \psi\end{array}}{\varphi \to \psi}\ \to\text{Intro}$$

Since the number of undischarged assumptions in δ_1 is $n-1$, the inductive hypothesis applies: there is a derivation δ_2 of $\varphi \to \psi$ from Γ. Since $\Gamma \vdash \varphi$ there is also a derivation δ_3 of φ from Γ. Now consider:

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots\, \delta_2 \\ \varphi \to \psi\end{array} \qquad \begin{array}{c}\Gamma \\ \vdots\, \delta_3 \\ \varphi\end{array}}{\psi}\;{\to}\text{Elim}$$

This shows $\Gamma \vdash \psi$. □

Proposition ntd.7. *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence φ.* *fol:ntd:ptn: prop:incons*

Proof. Exercise. □

Problem ntd.1. Prove Proposition ntd.7

Proposition ntd.8 (Compactness)**.** *fol:ntd:ptn: prop:proves-compact*

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*

2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof. 1. If $\Gamma \vdash \varphi$, then there is a derivation δ of φ from Γ. Let Γ_0 be the set of undischarged assumptions of δ. Since any derivation is finite, Γ_0 can only contain finitely many sentences. So, δ is a derivation of φ from a finite $\Gamma_0 \subseteq \Gamma$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \bot$.

□

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Bibliography