ntd.1 Proof-Theoretic Notions

This section collects the definitions the provability relation and consistency for natural deduction.

Just as we’ve defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding proof-theoretic notions. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain sentences from others. It was an important discovery that these notions coincide. That they do is the content of the soundness and completeness theorems.

Definition ntd.1 (Theorems). A sentence \( \varphi \) is a theorem if there is a derivation of \( \varphi \) in natural deduction in which all assumptions are discharged. We write \( \vdash \varphi \) if \( \varphi \) is a theorem and \( \nvdash \varphi \) if it is not.

Definition ntd.2 (Derivability). A sentence \( \varphi \) is derivable from a set of sentences \( \Gamma \), \( \Gamma \vdash \varphi \), if there is a derivation with conclusion \( \varphi \) and in which every assumption is either discharged or is in \( \Gamma \). If \( \varphi \) is not derivable from \( \Gamma \) we write \( \Gamma \nvdash \varphi \).

Definition ntd.3 (Consistency). A set of sentences \( \Gamma \) is inconsistent iff \( \Gamma \vdash \bot \). If \( \Gamma \) is not inconsistent, i.e., if \( \Gamma \nvdash \bot \), we say it is consistent.

Proposition ntd.4 (Reflexivity). If \( \varphi \in \Gamma \), then \( \Gamma \vdash \varphi \).

Proof. The assumption \( \varphi \) by itself is a derivation of \( \varphi \) where every undischarged assumption (i.e., \( \varphi \)) is in \( \Gamma \).

Proposition ntd.5 (Monotonicity). If \( \Gamma \subseteq \Delta \) and \( \Gamma \vdash \varphi \), then \( \Delta \vdash \varphi \).

Proof. Any derivation of \( \varphi \) from \( \Gamma \) is also a derivation of \( \varphi \) from \( \Delta \).

Proposition ntd.6 (Transitivity). If \( \Gamma \vdash \varphi \) and \( \{ \varphi \} \cup \Delta \vdash \psi \), then \( \Gamma \cup \Delta \vdash \psi \).

Proof. If \( \Gamma \vdash \varphi \), there is a derivation \( \delta_0 \) of \( \varphi \) with all undischarged assumptions in \( \Gamma \). If \( \{ \varphi \} \cup \Delta \vdash \psi \), then there is a derivation \( \delta_1 \) of \( \psi \) with all undischarged assumptions in \( \{ \varphi \} \cup \Delta \). Now consider:

\[
\Delta, [\varphi] \vdash \psi \\
\Delta \vdash \Gamma \\
\psi \vdash \psi \\
\varphi \rightarrow \psi \rightarrow \text{Intro} \\
\psi \rightarrow \text{Elim}
\]

proof-theoretic-notions rev: d4e99d0 (2023-10-08) by OLP / CC–BY
The undischarged assumptions are now all among $\Gamma \cup \Delta$, so this shows $\Gamma \cup \Delta \vdash \psi$.

When $\Gamma = \{ \varphi_1, \varphi_2, \ldots, \varphi_k \}$ is a finite set we may use the simplified notation $\varphi_1, \varphi_2, \ldots, \varphi_k \vdash \psi$ for $\Gamma \vdash \psi$, in particular $\varphi \vdash \psi$ means that $\{ \varphi \} \vdash \psi$.

Note that if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \ldots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each $i$, then $\Gamma \vdash \psi$.

**Proposition ntd.7.** The following are equivalent.

1. $\Gamma$ is inconsistent.
2. $\Gamma \vdash \varphi$ for every sentence $\varphi$.
3. $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg \varphi$ for some sentence $\varphi$.

**Proof.** Exercise.

**Problem ntd.1.** Prove Proposition ntd.7

**Proposition ntd.8 (Compactness).**

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of $\Gamma$ is consistent, then $\Gamma$ is consistent.

**Proof.** 1. If $\Gamma \vdash \varphi$, then there is a derivation $\delta$ of $\varphi$ from $\Gamma$. Let $\Gamma_0$ be the set of undischarged assumptions of $\delta$. Since any derivation is finite, $\Gamma_0$ can only contain finitely many sentences. So, $\delta$ is a derivation of $\varphi$ from a finite $\Gamma_0 \subseteq \Gamma$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \bot$.

**Photo Credits**

**Bibliography**