

ntd.1 Proof-Theoretic Notions

fol:ntd:ptn:
sec Just as we've defined a number of important semantic notions (validity, explanation entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the **derivability** or **non-derivability** of certain **sentences** from others. It was an important discovery, due to Gödel, that these notions coincide. That they do is the content of the *completeness theorem*.

Definition ntd.1 (Theorems). A **sentence** φ is a *theorem* if there is a **derivation** of φ in natural deduction in which all assumptions are **discharged**. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition ntd.2 (Derivability). A **sentence** φ is *derivable* from a set of **sentences** Γ , $\Gamma \vdash \varphi$, if there is a **derivation** with conclusion φ and in which every assumption is either **discharged** or is in Γ . If φ is not **derivable** from Γ we write $\Gamma \not\vdash \varphi$.

Definition ntd.3 (Consistency). A set of **sentences** Γ is *inconsistent* iff $\Gamma \vdash \perp$. If Γ is not inconsistent, i.e., if $\Gamma \not\vdash \perp$, we say it is *consistent*.

fol:ntd:ptn:
prop:reflexivity **Proposition ntd.4** (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. The assumption φ by itself is a **derivation** of φ where every **undischarged** assumption (i.e., φ) is in Γ . \square

fol:ntd:ptn:
prop:monotony **Proposition ntd.5** (Monotony). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any **derivation** of φ from Γ is also a **derivation** of φ from Δ . \square

fol:ntd:ptn:
prop:transitivity **Proposition ntd.6** (Transitivity). *If $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$ and $\Delta \vdash \psi$, then $\Gamma \vdash \psi$.*

Proof. If $\Delta \vdash \psi$, then there is a **derivation** δ_0 of ψ with all **undischarged** assumptions in Δ . We show that $\Gamma \vdash \psi$ by induction on the number n of **undischarged** assumptions in δ_0 .

If $n = 0$, then δ_0 has no **undischarged** assumptions, and so also counts as a **derivation** of ψ from Γ .

Otherwise, pick an **undischarged** assumption φ in δ_0 and let Δ_1 be the remaining **undischarged** assumptions. We obtain the **derivation** δ_1 :

$$\frac{\begin{array}{c} \Delta_1, [\varphi]^1 \\ \vdots \\ \delta_0 \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \rightarrow\text{Intro}$$

Since the number of **undischarged** assumptions in δ_1 is $n - 1$, the inductive hypothesis applies: there is a **derivation** δ_2 of $\varphi \rightarrow \psi$ from Γ . Since $\Gamma \vdash \varphi$ there is also a **derivation** δ_3 of φ from Γ . Now consider:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \delta_2 \\ \vdots \\ \varphi \rightarrow \psi \end{array} \quad \begin{array}{c} \Gamma \\ \vdots \\ \delta_3 \\ \vdots \\ \varphi \end{array}}{\psi} \rightarrow\text{Elim}$$

This shows $\Gamma \vdash \psi$. □

Proposition ntd.7. Γ is inconsistent iff $\Gamma \vdash \varphi$ for every **sentence** φ . *fol.ntd.ptn:*
prop:incons

Proof. Exercise. □

Problem ntd.1. Prove [Proposition ntd.7](#)

Proposition ntd.8 (Compactness). *fol.ntd.ptn:*
prop:proves-compact

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of Γ is consistent, then Γ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a **derivation** δ of φ from Γ . Let Γ_0 be the set of **undischarged** assumptions of δ . Since any **derivation** is finite, Γ_0 can only contain finitely many **sentences**. So, δ is a **derivation** of φ from a finite $\Gamma_0 \subseteq \Gamma$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \perp$. □

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Bibliography