Chapter udf

Natural Deduction

This chapter presents a natural deduction system in the style of Gentzen/Prawitz.

To include or exclude material relevant to natural deduction as a proof system, use the "prfND" tag.

ntd.1 Rules and Derivations

fol:ntd:rul: sec Natural deduction systems are meant to closely parallel the informal reasoning used in mathematical proof (hence it is somewhat "natural"). Natural deduction proofs begin with assumptions. Inference rules are then applied. Assumptions are "discharged" by the ¬Intro, →Intro, ∨Elim and ∃Elim inference rules, and the label of the discharged assumption is placed beside the inference for clarity. explanation

Definition ntd.1 (Assumption). An *assumption* is any sentence in the topmost position of any branch.

Derivations in natural deduction are certain trees of sentences, where the topmost sentences are assumptions, and if a sentence stands below one, two, or three other sequents, it must follow correctly by a rule of inference. The sentences at the top of the inference are called the *premises* and the sentence below the *conclusion* of the inference. The rules come in pairs, an introduction and an elimination rule for each logical operator. They introduce a logical operator in the conclusion or remove a logical operator from a premise of the rule. Some of the rules allow an assumption of a certain type to be *discharged*. To indicate which assumption is discharged by which inference, we also assign labels to both the assumption and the inference. This is indicated by writing the assumption as "$[\varphi]^n$."

It is customary to consider rules for all the logical operators $\land$, $\lor$, $\rightarrow$, $\neg$, and $\bot$, even if some of those are defined.

ntd.2 Propositional Rules

Rules for $\land$ fol:ntd:prl:sec

$$\frac{\varphi \qquad \psi}{\varphi \land \psi}\land\text{Intro}$$

$$\frac{\varphi \land \psi}{\varphi}\land\text{Elim} \qquad \frac{\varphi \land \psi}{\psi}\land\text{Elim}$$

Rules for $\lor$

$$\frac{\varphi}{\varphi \lor \psi}\lor\text{Intro} \qquad \frac{\psi}{\varphi \lor \psi}\lor\text{Intro}$$

$$\begin{array}{ccc} & [\varphi]^n & [\psi]^n \\ & \vdots & \vdots \end{array}$$
$$n\,\frac{\varphi \lor \psi \qquad \chi \qquad \chi}{\chi}\lor\text{Elim}$$

Rules for $\to$

$$\begin{array}{c} [\varphi]^n \\ \vdots \end{array}$$
$$n\,\frac{\psi}{\varphi \to \psi}\to\text{Intro}$$

$$\frac{\varphi \to \psi \qquad \varphi}{\psi}\to\text{Elim}$$

Rules for $\neg$

$$\begin{array}{c} [\varphi]^n \\ \vdots \end{array}$$
$$n\,\frac{\bot}{\neg\varphi}\neg\text{Intro}$$

$$\frac{\neg\varphi \qquad \varphi}{\bot}\neg\text{Elim}$$

Rules for $\bot$

$$\frac{\bot}{\varphi}\ \bot_I \qquad\qquad \begin{array}{c}[\neg\varphi]^n\\ \vdots\\ {}_n\dfrac{\bot}{\varphi}\ \bot_C\end{array}$$

Note that $\neg$Intro and $\bot_C$ are very similar: The difference is that $\neg$Intro derives a negated sentence $\neg\varphi$ but $\bot_C$ a positive sentence φ.

Whenever a rule indicates that some assumption may be discharged, we take this to be a permission, but not a requirement. E.g., in the $\to$Intro rule, we may discharge any number of assumptions of the form φ in the derivation of the premise ψ, including zero.

ntd.3 Quantifier Rules

fol:ntd:qrl:sec

Rules for $\forall$

$$\frac{\varphi(a)}{\forall x\,\varphi(x)}\ \forall\text{Intro} \qquad\qquad \frac{\forall x\,\varphi(x)}{\varphi(t)}\ \forall\text{Elim}$$

In the rules for $\forall$, t is a closed term (a term that does not contain any variables), and a is a constant symbol which does not occur in the conclusion $\forall x\,\varphi(x)$, or in any assumption which is undischarged in the derivation ending with the premise $\varphi(a)$. We call a the *eigenvariable* of the $\forall$Intro inference.[1]

Rules for $\exists$

$$\frac{\varphi(t)}{\exists x\,\varphi(x)}\ \exists\text{Intro} \qquad\qquad {}_n\dfrac{\exists x\,\varphi(x) \qquad \begin{array}{c}[\varphi(a)]^n\\ \vdots\\ \chi\end{array}}{\chi}\ \exists\text{Elim}$$

Again, t is a closed term, and a is a constant which does not occur in the premise $\exists x\,\varphi(x)$, in the conclusion χ, or any assumption which is undischarged

[1] We use the term "eigenvariable" even though a in the above rule is a constant. This has historical reasons.

in the derivations ending with the two premises (other than the assumptions $\varphi(a)$). We call a the *eigenvariable* of the $\exists$Elim inference.

The condition that an eigenvariable neither occur in the premises nor in any assumption that is undischarged in the derivations leading to the premises for the $\forall$Intro or $\exists$Elim inference is called the *eigenvariable condition*.

explanation Recall the convention that when φ is a formula with the variable x free, we indicate this by writing $\varphi(x)$. In the same context, $\varphi(t)$ then is short for $\varphi[t/x]$. So we could also write the $\exists$Intro rule as:

$$\frac{\varphi[t/x]}{\exists x\,\varphi}\;\exists\text{Intro}$$

Note that t may already occur in φ, e.g., φ might be $P(t,x)$. Thus, inferring $\exists x\,P(t,x)$ from $P(t,t)$ is a correct application of $\exists$Intro—you may "replace" one or more, and not necessarily all, occurrences of t in the premise by the bound variable x. However, the eigenvariable conditions in $\forall$Intro and $\exists$Elim require that the constant symbol a does not occur in φ. So, you cannot correctly infer $\forall x\,P(a,x)$ from $P(a,a)$ using $\forall$Intro.

explanation In $\exists$Intro and $\forall$Elim there are no restrictions, and the term t can be anything, so we do not have to worry about any conditions. On the other hand, in the $\exists$Elim and $\forall$Intro rules, the eigenvariable condition requires that the constant symbol a does not occur anywhere in the conclusion or in an undischarged assumption. The condition is necessary to ensure that the system is sound, i.e., only derives sentences from undischarged assumptions from which they follow. Without this condition, the following would be allowed:

$$\frac{\exists x\,\varphi(x) \qquad \dfrac{[\varphi(a)]^1}{\forall x\,\varphi(x)}\;{*}\forall\text{Intro}}{\forall x\,\varphi(x)}\;\exists\text{Elim}$$

However, $\exists x\,\varphi(x) \nvDash \forall x\,\varphi(x)$.

As the elimination rules for quantifiers only allow substituting closed terms for variables, it follows that any formula that can be derived from a set of sentences is itself a sentence.

ntd.4 Derivations

explanation We've said what an assumption is, and we've given the rules of inference. Derivations in natural deduction are inductively generated from these: each derivation either is an assumption on its own, or consists of one, two, or three derivations followed by a correct inference. fol:ntd:der:sec

Definition ntd.2 (Derivation). A *derivation* of a sentence φ from assumptions Γ is a finite tree of sentences satisfying the following conditions:

1. The topmost sentences of the tree are either in Γ or are discharged by an inference in the tree.

2. The bottommost sentence of the tree is φ.

3. Every sentence in the tree except the sentence φ at the bottom is a premise of a correct application of an inference rule whose conclusion stands directly below that sentence in the tree.

We then say that φ is the *conclusion* of the derivation and Γ its undischarged assumptions.

If a derivation of φ from Γ exists, we say that φ is *derivable* from Γ, or in symbols: $\Gamma \vdash \varphi$. If there is a derivation of φ in which every assumption is discharged, we write $\vdash \varphi$.

Example ntd.3. Every assumption on its own is a derivation. So, e.g., φ by itself is a derivation, and so is ψ by itself. We can obtain a new derivation from these by applying, say, the $\wedge$Intro rule,

$$\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}\ \wedge\text{Intro}$$

These rules are meant to be general: we can replace the φ and ψ in it with any sentences, e.g., by χ and θ. Then the conclusion would be $\chi \wedge \theta$, and so

$$\dfrac{\chi \qquad \theta}{\chi \wedge \theta}\ \wedge\text{Intro}$$

is a correct derivation. Of course, we can also switch the assumptions, so that θ plays the role of φ and χ that of ψ. Thus,

$$\dfrac{\theta \qquad \chi}{\theta \wedge \chi}\ \wedge\text{Intro}$$

is also a correct derivation.

We can now apply another rule, say, $\to$Intro, which allows us to conclude a conditional and allows us to discharge any assumption that is identical to the antecedent of that conditional. So both of the following would be correct derivations:

$$1\ \dfrac{\dfrac{[\chi]^1 \qquad \theta}{\chi \wedge \theta}\ \wedge\text{Intro}}{\chi \to (\chi \wedge \theta)}\ \to\text{Intro} \qquad 1\ \dfrac{\dfrac{\chi \qquad [\theta]^1}{\chi \wedge \theta}\ \wedge\text{Intro}}{\theta \to (\chi \wedge \theta)}\ \to\text{Intro}$$

They show, respectively, that $\theta \vdash \chi \to (\chi \wedge \theta)$ and $\chi \vdash \theta \to (\chi \wedge \theta)$.

Remember that discharging of assumptions is a permission, not a requirement: we don't have to discharge the assumptions. In particular, we can apply a rule even if the assumptions are not present in the derivation. For instance, the following is legal, even though there is no assumption φ to be discharged:

$$1\ \dfrac{\psi}{\varphi \to \psi}\ \to\text{Intro}$$

ntd.5 Examples of Derivations

fol:ntd:pro: sec

Example ntd.4. Let's give a derivation of the sentence $(\varphi \land \psi) \to \varphi$.

We begin by writing the desired conclusion at the bottom of the derivation.

$$\frac{}{(\varphi \land \psi) \to \varphi}$$

Next, we need to figure out what kind of inference could result in a sentence of this form. The main operator of the conclusion is $\to$, so we'll try to arrive at the conclusion using the $\to$Intro rule. It is best to write down the assumptions involved and label the inference rules as you progress, so it is easy to see whether all assumptions have been discharged at the end of the proof.

$$\begin{array}{c} [\varphi \land \psi]^1 \\ \vdots \\ {}_1\,\dfrac{\varphi}{(\varphi \land \psi) \to \varphi}\,{\to}\mathrm{Intro} \end{array}$$

We now need to fill in the steps from the assumption $\varphi \land \psi$ to φ. Since we only have one connective to deal with, $\land$, we must use the $\land$ elim rule. This gives us the following proof:

$${}_1\,\dfrac{\dfrac{[\varphi \land \psi]^1}{\varphi}\,{\land}\mathrm{Elim}}{(\varphi \land \psi) \to \varphi}\,{\to}\mathrm{Intro}$$

We now have a correct derivation of $(\varphi \land \psi) \to \varphi$.

Example ntd.5. Now let's give a derivation of $(\neg\varphi \lor \psi) \to (\varphi \to \psi)$.

We begin by writing the desired conclusion at the bottom of the derivation.

$$\frac{}{(\neg\varphi \lor \psi) \to (\varphi \to \psi)}$$

To find a logical rule that could give us this conclusion, we look at the logical connectives in the conclusion: $\neg$, $\lor$, and $\to$. We only care at the moment about the first occurrence of $\to$ because it is the main operator of the sentence in the end-sequent, while $\neg$, $\lor$ and the second occurrence of $\to$ are inside the scope of another connective, so we will take care of those later. We therefore start with the $\to$Intro rule. A correct application must look like this:

$$\begin{array}{c} [\neg\varphi \lor \psi]^1 \\ \vdots \\ {}_1\,\dfrac{\varphi \to \psi}{(\neg\varphi \lor \psi) \to (\varphi \to \psi)}\,{\to}\mathrm{Intro} \end{array}$$

This leaves us with two possibilities to continue. Either we can keep working from the bottom up and look for another application of the $\rightarrow$Intro rule, or we can work from the top down and apply a $\lor$Elim rule. Let us apply the latter. We will use the assumption $\lnot\varphi \lor \psi$ as the leftmost premise of $\lor$Elim. For a valid application of $\lor$Elim, the other two premises must be identical to the conclusion $\varphi \rightarrow \psi$, but each may be derived in turn from another assumption, namely one of the two disjuncts of $\lnot\varphi \lor \psi$. So our derivation will look like this:

$$
\begin{prooftree}
\AxiomC{$[\lnot\varphi \lor \psi]^1$}
\AxiomC{$[\lnot\varphi]^2$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\varphi \rightarrow \psi$}
\AxiomC{$[\psi]^2$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\varphi \rightarrow \psi$}
\LeftLabel{2}
\RightLabel{$\lor$Elim}
\TrinaryInfC{$\varphi \rightarrow \psi$}
\LeftLabel{1}
\RightLabel{$\rightarrow$Intro}
\UnaryInfC{$(\lnot\varphi \lor \psi) \rightarrow (\varphi \rightarrow \psi)$}
\end{prooftree}
$$

In each of the two branches on the right, we want to derive $\varphi \rightarrow \psi$, which is best done using $\rightarrow$Intro.

$$
\begin{prooftree}
\AxiomC{$[\lnot\varphi \lor \psi]^1$}
\AxiomC{$[\lnot\varphi]^2, [\varphi]^3$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{ψ}
\LeftLabel{3}
\RightLabel{$\rightarrow$Intro}
\UnaryInfC{$\varphi \rightarrow \psi$}
\AxiomC{$[\psi]^2, [\varphi]^4$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{ψ}
\LeftLabel{4}
\RightLabel{$\rightarrow$Intro}
\UnaryInfC{$\varphi \rightarrow \psi$}
\LeftLabel{2}
\RightLabel{$\lor$Elim}
\TrinaryInfC{$\varphi \rightarrow \psi$}
\LeftLabel{1}
\RightLabel{$\rightarrow$Intro}
\UnaryInfC{$(\lnot\varphi \lor \psi) \rightarrow (\varphi \rightarrow \psi)$}
\end{prooftree}
$$

For the two missing parts of the derivation, we need derivations of ψ from $\lnot\varphi$ and φ in the middle, and from φ and ψ on the left. Let's take the former first. $\lnot\varphi$ and φ are the two premises of $\lnot$Elim:

$$
\begin{prooftree}
\AxiomC{$[\lnot\varphi]^2$}
\AxiomC{$[\varphi]^3$}
\RightLabel{$\lnot$Elim}
\BinaryInfC{$\bot$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{ψ}
\end{prooftree}
$$

By using $\bot_I$, we can obtain ψ as a conclusion and complete the branch.

$$
\begin{prooftree}
\AxiomC{$[\lnot\varphi \lor \psi]^1$}
\AxiomC{$[\lnot\varphi]^2$}
\AxiomC{$[\varphi]^3$}
\RightLabel{$\bot$Intro}
\BinaryInfC{$\bot$}
\RightLabel{$\bot_I$}
\UnaryInfC{ψ}
\LeftLabel{3}
\RightLabel{$\rightarrow$Intro}
\UnaryInfC{$\varphi \rightarrow \psi$}
\AxiomC{$[\psi]^2, [\varphi]^4$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{ψ}
\LeftLabel{4}
\RightLabel{$\rightarrow$Intro}
\UnaryInfC{$\varphi \rightarrow \psi$}
\LeftLabel{2}
\RightLabel{$\lor$Elim}
\TrinaryInfC{$\varphi \rightarrow \psi$}
\LeftLabel{1}
\RightLabel{$\rightarrow$Intro}
\UnaryInfC{$(\lnot\varphi \lor \psi) \rightarrow (\varphi \rightarrow \psi)$}
\end{prooftree}
$$

Let's now look at the rightmost branch. Here it's important to realize that the definition of derivation *allows assumptions to be discharged* but *does not require* them to be. In other words, if we can derive ψ from one of the assumptions φ and ψ without using the other, that's ok. And to derive ψ from ψ is trivial: ψ by itself is such a derivation, and no inferences are needed. So we can simply delete the assumption φ.

$$
{}^{1}\dfrac{{}^{2}\dfrac{[\neg\varphi \lor \psi]^1 \qquad {}^{3}\dfrac{\dfrac{\dfrac{[\neg\varphi]^2 \qquad [\varphi]^3}{\bot}\,\neg\mathrm{Elim}}{\psi}\,\bot_I}{\varphi \to \psi}\,{\to}\mathrm{Intro} \qquad \dfrac{[\psi]^2}{\varphi \to \psi}\,{\to}\mathrm{Intro}}{\varphi \to \psi}\,{\lor}\mathrm{Elim}}{(\neg\varphi \lor \psi) \to (\varphi \to \psi)}\,{\to}\mathrm{Intro}
$$

Note that in the finished derivation, the rightmost $\to$Intro inference does not actually discharge any assumptions.

Example ntd.6. So far we have not needed the $\bot_C$ rule. It is special in that it allows us to discharge an assumption that isn't a sub-formula of the conclusion of the rule. It is closely related to the $\bot_I$ rule. In fact, the $\bot_I$ rule is a special case of the $\bot_C$ rule—there is a logic called "intuitionistic logic" in which only $\bot_I$ is allowed. The $\bot_C$ rule is a last resort when nothing else works. For instance, suppose we want to derive $\varphi \lor \neg\varphi$. Our usual strategy would be to attempt to derive $\varphi \lor \neg\varphi$ using $\lor$Intro. But this would require us to derive either φ or $\neg\varphi$ from no assumptions, and this can't be done. $\bot_C$ to the rescue!

$$
\begin{array}{c}
[\neg(\varphi \lor \neg\varphi)]^1 \\
\vdots \\
{}^{1}\dfrac{\bot}{\varphi \lor \neg\varphi}\,\bot_C
\end{array}
$$

Now we're looking for a derivation of $\bot$ from $\neg(\varphi \lor \neg\varphi)$. Since $\bot$ is the conclusion of $\neg$Elim we might try that:

$$
{}^{1}\dfrac{\dfrac{\begin{array}{c}[\neg(\varphi \lor \neg\varphi)]^1 \\ \vdots \\ \neg\varphi\end{array} \qquad \begin{array}{c}[\neg(\varphi \lor \neg\varphi)]^1 \\ \vdots \\ \varphi\end{array}}{\bot}\,\neg\mathrm{Elim}}{\varphi \lor \neg\varphi}\,\bot_C
$$

Our strategy for finding a derivation of $\neg\varphi$ calls for an application of $\neg$Intro:

$$
\begin{prooftree}
\AxiomC{$[\neg(\varphi \lor \neg\varphi)]^1, [\varphi]^2$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{$\neg$Intro}
\UnaryInfC{$\neg\varphi$}
\AxiomC{$[\neg(\varphi \lor \neg\varphi)]^1$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{φ}
\RightLabel{$\neg$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\bot_C$}
\UnaryInfC{$\varphi \lor \neg\varphi$}
\end{prooftree}
$$

Here, we can get $\bot$ easily by applying $\neg$Elim to the assumption $\neg(\varphi \lor \neg\varphi)$ and $\varphi \lor \neg\varphi$ which follows from our new assumption φ by $\lor$Intro:

$$
\begin{prooftree}
\AxiomC{$[\neg(\varphi \lor \neg\varphi)]^1$}
\AxiomC{$[\varphi]^2$}
\RightLabel{$\lor$Intro}
\UnaryInfC{$\varphi \lor \neg\varphi$}
\RightLabel{$\neg$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{$\neg$Intro}
\UnaryInfC{$\neg\varphi$}
\AxiomC{$[\neg(\varphi \lor \neg\varphi)]^1$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{φ}
\RightLabel{$\neg$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\bot_C$}
\UnaryInfC{$\varphi \lor \neg\varphi$}
\end{prooftree}
$$

On the right side we use the same strategy, except we get φ by $\bot_C$:

$$
\begin{prooftree}
\AxiomC{$[\neg(\varphi \lor \neg\varphi)]^1$}
\AxiomC{$[\varphi]^2$}
\RightLabel{$\lor$Intro}
\UnaryInfC{$\varphi \lor \neg\varphi$}
\RightLabel{$\neg$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{$\neg$Intro}
\UnaryInfC{$\neg\varphi$}
\AxiomC{$[\neg(\varphi \lor \neg\varphi)]^1$}
\AxiomC{$[\neg\varphi]^3$}
\RightLabel{$\lor$Intro}
\UnaryInfC{$\varphi \lor \neg\varphi$}
\RightLabel{$\neg$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{3}
\RightLabel{$\bot_C$}
\UnaryInfC{φ}
\RightLabel{$\neg$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\bot_C$}
\UnaryInfC{$\varphi \lor \neg\varphi$}
\end{prooftree}
$$

Problem ntd.1. Give derivations that show the following:

1. $\varphi \land (\psi \land \chi) \vdash (\varphi \land \psi) \land \chi$.
2. $\varphi \lor (\psi \lor \chi) \vdash (\varphi \lor \psi) \lor \chi$.
3. $\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi)$.
4. $\varphi \vdash \neg\neg\varphi$.

Problem ntd.2. Give derivations that show the following:

1. $(\varphi \lor \psi) \to \chi \vdash \varphi \to \chi$.
2. $(\varphi \to \chi) \land (\psi \to \chi) \vdash (\varphi \lor \psi) \to \chi$.
3. $\vdash \neg(\varphi \land \neg\varphi)$.
4. $\psi \to \varphi \vdash \neg\varphi \to \neg\psi$.
5. $\vdash (\varphi \to \neg\varphi) \to \neg\varphi$.
6. $\vdash \neg(\varphi \to \psi) \to \neg\psi$.

7. $\varphi \to \chi \vdash \neg(\varphi \land \neg\chi)$.

8. $\varphi \land \neg\chi \vdash \neg(\varphi \to \chi)$.

9. $\varphi \lor \psi, \neg\psi \vdash \varphi$.

10. $\neg\varphi \lor \neg\psi \vdash \neg(\varphi \land \psi)$.

11. $\vdash (\neg\varphi \land \neg\psi) \to \neg(\varphi \lor \psi)$.

12. $\vdash \neg(\varphi \lor \psi) \to (\neg\varphi \land \neg\psi)$.

Problem ntd.3. Give derivations that show the following:

1. $\neg(\varphi \to \psi) \vdash \varphi$.

2. $\neg(\varphi \land \psi) \vdash \neg\varphi \lor \neg\psi$.

3. $\varphi \to \psi \vdash \neg\varphi \lor \psi$.

4. $\vdash \neg\neg\varphi \to \varphi$.

5. $\varphi \to \psi, \neg\varphi \to \psi \vdash \psi$.

6. $(\varphi \land \psi) \to \chi \vdash (\varphi \to \chi) \lor (\psi \to \chi)$.

7. $(\varphi \to \psi) \to \varphi \vdash \varphi$.

8. $\vdash (\varphi \to \psi) \lor (\psi \to \chi)$.

(These all require the $\bot_C$ rule.)

ntd.6 Derivations with Quantifiers

fol:ntd:prq:sec

Example ntd.7. When dealing with quantifiers, we have to make sure not to violate the eigenvariable condition, and sometimes this requires us to play around with the order of carrying out certain inferences. In general, it helps to try and take care of rules subject to the eigenvariable condition first (they will be lower down in the finished proof).

Let's see how we'd give a derivation of the formula $\exists x\, \neg\varphi(x) \to \neg\forall x\, \varphi(x)$. Starting as usual, we write

$$\frac{}{\exists x\, \neg\varphi(x) \to \neg\forall x\, \varphi(x)}$$

We start by writing down what it would take to justify that last step using the $\to$Intro rule.

$$
\cfrac{\begin{matrix}[\exists x\,\neg\varphi(x)]^1\\ \vdots\\ \neg\forall x\,\varphi(x)\end{matrix}}{\exists x\,\neg\varphi(x) \rightarrow \neg\forall x\,\varphi(x)}\;{\rightarrow}\mathrm{Intro}^{1}
$$

Since there is no obvious rule to apply to $\neg\forall x\,\varphi(x)$, we will proceed by setting up the derivation so we can use the $\exists$Elim rule. Here we must pay attention to the eigenvariable condition, and choose a constant that does not appear in $\exists x\,\varphi(x)$ or any assumptions that it depends on. (Since no constant symbols appear, however, any choice will do fine.)

$$
\cfrac{\cfrac{[\exists x\,\neg\varphi(x)]^1 \qquad \begin{matrix}[\neg\varphi(a)]^2\\ \vdots\\ \neg\forall x\,\varphi(x)\end{matrix}}{\neg\forall x\,\varphi(x)}\;\exists\mathrm{Elim}^{2}}{\exists x\,\neg\varphi(x) \rightarrow \neg\forall x\,\varphi(x)}\;{\rightarrow}\mathrm{Intro}^{1}
$$

In order to derive $\neg\forall x\,\varphi(x)$, we will attempt to use the $\neg$Intro rule: this requires that we derive a contradiction, possibly using $\forall x\,\varphi(x)$ as an additional assumption. Of course, this contradiction may involve the assumption $\neg\varphi(a)$ which will be discharged by the $\exists$Elim inference. We can set it up as follows:

$$
\cfrac{\cfrac{[\exists x\,\neg\varphi(x)]^1 \qquad \cfrac{\begin{matrix}[\neg\varphi(a)]^2, [\forall x\,\varphi(x)]^3\\ \vdots\\ \bot\end{matrix}}{\neg\forall x\,\varphi(x)}\;\neg\mathrm{Intro}^{3}}{\neg\forall x\,\varphi(x)}\;\exists\mathrm{Elim}^{2}}{\exists x\,\neg\varphi(x) \rightarrow \neg\forall x\,\varphi(x)}\;{\rightarrow}\mathrm{Intro}^{1}
$$

It looks like we are close to getting a contradiction. The easiest rule to apply is the $\forall$Elim, which has no eigenvariable conditions. Since we can use any term we want to replace the universally quantified x, it makes the most sense to continue using a so we can reach a contradiction.

$$
\cfrac{\cfrac{[\exists x\,\neg\varphi(x)]^1 \qquad \cfrac{\cfrac{[\neg\varphi(a)]^2 \qquad \cfrac{[\forall x\,\varphi(x)]^3}{\varphi(a)}\;\forall\mathrm{Elim}}{\bot}\;\neg\mathrm{Elim}}{\neg\forall x\,\varphi(x)}\;\neg\mathrm{Intro}^{3}}{\neg\forall x\,\varphi(x)}\;\exists\mathrm{Elim}^{2}}{\exists x\,\neg\varphi(x) \rightarrow \neg\forall x\,\varphi(x)}\;{\rightarrow}\mathrm{Intro}^{1}
$$

It is important, especially when dealing with quantifiers, to double check at this point that the eigenvariable condition has not been violated. Since the only rule we applied that is subject to the eigenvariable condition was $\exists$Elim, and the eigenvariable a does not occur in any assumptions it depends on, this is a correct derivation.

Example ntd.8. Sometimes we may derive a formula from other formulas. In these cases, we may have undischarged assumptions. It is important to keep track of our assumptions as well as the end goal.

Let's see how we'd give a derivation of the formula $\exists x\, \chi(x,b)$ from the assumptions $\exists x\, (\varphi(x) \land \psi(x))$ and $\forall x\, (\psi(x) \to \chi(x,b))$. Starting as usual, we write the conclusion at the bottom.

$$\frac{}{\exists x\, \chi(x,b)}$$

We have two premises to work with. To use the first, i.e., try to find a derivation of $\exists x\, \chi(x,b)$ from $\exists x\, (\varphi(x) \land \psi(x))$ we would use the $\exists$Elim rule. Since it has an eigenvariable condition, we will apply that rule first. We get the following:

$$\begin{array}{c} \\ \\ \\ \end{array}{}_{1}\,\dfrac{\exists x\, (\varphi(x) \land \psi(x)) \qquad \begin{array}{c}[\varphi(a) \land \psi(a)]^1 \\ \vdots \\ \exists x\, \chi(x,b)\end{array}}{\exists x\, \chi(x,b)}\;\exists\text{Elim}$$

The two assumptions we are working with share ψ. It may be useful at this point to apply $\land$Elim to separate out $\psi(a)$.

$${}_{1}\,\dfrac{\exists x\, (\varphi(x) \land \psi(x)) \qquad \begin{array}{c}\dfrac{[\varphi(a) \land \psi(a)]^1}{\psi(a)}\;\land\text{Elim} \\ \vdots \\ \exists x\, \chi(x,b)\end{array}}{\exists x\, \chi(x,b)}\;\exists\text{Elim}$$

The second assumption we have to work with is $\forall x\, (\psi(x) \to \chi(x,b))$. Since there is no eigenvariable condition we can instantiate x with the constant symbol a using $\forall$Elim to get $\psi(a) \to \chi(a,b)$. We now have both $\psi(a) \to \chi(a,b)$ and $\psi(a)$. Our next move should be a straightforward application of the $\to$Elim rule.

$$
\begin{prooftree}
\AxiomC{$\exists x\,(\varphi(x) \land \psi(x))$}
\AxiomC{$\forall x\,(\psi(x) \to \chi(x,b))$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\psi(a) \to \chi(a,b)$}
\AxiomC{$[\varphi(a) \land \psi(a)]^1$}
\RightLabel{$\land$Elim}
\UnaryInfC{$\psi(a)$}
\RightLabel{$\to$Elim}
\BinaryInfC{$\chi(a,b)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\exists x\,\chi(x,b)$}
\LeftLabel{1}
\RightLabel{$\exists$Elim}
\BinaryInfC{$\exists x\,\chi(x,b)$}
\end{prooftree}
$$

We are so close! One application of ∃Intro and we have reached our goal.

$$
\begin{prooftree}
\AxiomC{$\exists x\,(\varphi(x) \land \psi(x))$}
\AxiomC{$\forall x\,(\psi(x) \to \chi(x,b))$}
\RightLabel{$\forall$Elim}
\UnaryInfC{$\psi(a) \to \chi(a,b)$}
\AxiomC{$[\varphi(a) \land \psi(a)]^1$}
\RightLabel{$\land$Elim}
\UnaryInfC{$\psi(a)$}
\RightLabel{$\to$Elim}
\BinaryInfC{$\chi(a,b)$}
\RightLabel{$\exists$Intro}
\UnaryInfC{$\exists x\,\chi(x,b)$}
\LeftLabel{1}
\RightLabel{$\exists$Elim}
\BinaryInfC{$\exists x\,\chi(x,b)$}
\end{prooftree}
$$

Since we ensured at each step that the eigenvariable conditions were not violated, we can be confident that this is a correct derivation.

Example ntd.9. Give a derivation of the formula $\lnot\forall x\,\varphi(x)$ from the assumptions $\forall x\,\varphi(x) \to \exists y\,\psi(y)$ and $\lnot\exists y\,\psi(y)$. Starting as usual, we write the target formula at the bottom.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\lnot\forall x\,\varphi(x)$}
\end{prooftree}
$$

The last line of the derivation is a negation, so let's try using ¬Intro. This will require that we figure out how to derive a contradiction.

$$
\begin{prooftree}
\AxiomC{$[\forall x\,\varphi(x)]^1$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\lnot$Intro}
\UnaryInfC{$\lnot\forall x\,\varphi(x)$}
\end{prooftree}
$$

So far so good. We can use ∀Elim but it's not obvious if that will help us get to our goal. Instead, let's use one of our assumptions. $\forall x\,\varphi(x) \to \exists y\,\psi(y)$ together with $\forall x\,\varphi(x)$ will allow us to use the →Elim rule.

$$
\begin{prooftree}
\AxiomC{$\forall x\,\varphi(x) \to \exists y\,\psi(y)$}
\AxiomC{$[\forall x\,\varphi(x)]^1$}
\RightLabel{$\to$Elim}
\BinaryInfC{$\exists y\,\psi(y)$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\lnot$Intro}
\UnaryInfC{$\lnot\forall x\,\varphi(x)$}
\end{prooftree}
$$

We now have one final assumption to work with, and it looks like this will help us reach a contradiction by using $\neg$Elim.

$$
{}^{1}\,\dfrac{\dfrac{\neg\exists y\,\psi(y) \qquad \dfrac{\forall x\,\varphi(x) \to \exists y\,\psi(y) \qquad [\forall x\,\varphi(x)]^1}{\exists y\,\psi(y)}\,{\to}\mathrm{Elim}}{\bot}\,\neg\mathrm{Elim}}{\neg\forall x\,\varphi(x)}\,\neg\mathrm{Intro}
$$

Problem ntd.4. Give derivations that show the following:

1. $\vdash (\forall x\,\varphi(x) \land \forall y\,\psi(y)) \to \forall z\,(\varphi(z) \land \psi(z))$.
2. $\vdash (\exists x\,\varphi(x) \lor \exists y\,\psi(y)) \to \exists z\,(\varphi(z) \lor \psi(z))$.
3. $\forall x\,(\varphi(x) \to \psi) \vdash \exists y\,\varphi(y) \to \psi$.
4. $\forall x\,\neg\varphi(x) \vdash \neg\exists x\,\varphi(x)$.
5. $\vdash \neg\exists x\,\varphi(x) \to \forall x\,\neg\varphi(x)$.
6. $\vdash \neg\exists x\,\forall y\,((\varphi(x,y) \to \neg\varphi(y,y)) \land (\neg\varphi(y,y) \to \varphi(x,y)))$.

Problem ntd.5. Give derivations that show the following:

1. $\vdash \neg\forall x\,\varphi(x) \to \exists x\,\neg\varphi(x)$.
2. $(\forall x\,\varphi(x) \to \psi) \vdash \exists y\,(\varphi(y) \to \psi)$.
3. $\vdash \exists x\,(\varphi(x) \to \forall y\,\varphi(y))$.

(These all require the $\bot_C$ rule.)

ntd.7 Proof-Theoretic Notions

fol:ntd:ptn: sec

> This section collects the definitions the provability relation and consistency for natural deduction.

explanation Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain sentences from others. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*.

Definition ntd.10 (Theorems). A sentence φ is a *theorem* if there is a derivation of φ in natural deduction in which all assumptions are discharged. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Definition ntd.11 (Derivability). A sentence φ is *derivable from* a set of sentences Γ, $\Gamma \vdash \varphi$, if there is a derivation with conclusion φ and in which every assumption is either discharged or is in Γ. If φ is not derivable from Γ we write $\Gamma \nvdash \varphi$.

Definition ntd.12 (Consistency). A set of sentences Γ is *inconsistent* iff $\Gamma \vdash \bot$. If Γ is not inconsistent, i.e., if $\Gamma \nvdash \bot$, we say it is *consistent*.

fol:ntd:ptn: prop:reflexivity

Proposition ntd.13 (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. The assumption φ by itself is a derivation of φ where every undischarged assumption (i.e., φ) is in Γ. □

fol:ntd:ptn: prop:monotonicity

Proposition ntd.14 (Monotonicity). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any derivation of φ from Γ is also a derivation of φ from Δ. □

fol:ntd:ptn: prop:transitivity

Proposition ntd.15 (Transitivity). *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

Proof. If $\Gamma \vdash \varphi$, there is a derivation δ_0 of φ with all undischarged assumptions in Γ. If $\{\varphi\} \cup \Delta \vdash \psi$, then there is a derivation δ_1 of ψ with all undischarged assumptions in $\{\varphi\} \cup \Delta$. Now consider:

$$\dfrac{{}^{1}\dfrac{\begin{matrix}\Delta, [\varphi]^1 \\ \vdots\ \delta_1 \\ \psi\end{matrix}}{\varphi \to \psi}\,{\to}\mathrm{Intro} \qquad \begin{matrix}\Gamma \\ \vdots\ \delta_0 \\ \varphi\end{matrix}}{\psi}\,{\to}\mathrm{Elim}$$

The undischarged assumptions are now all among $\Gamma \cup \Delta$, so this shows $\Gamma \cup \Delta \vdash \psi$. □

When $\Gamma = \{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ is a finite set we may use the simplified notation $\varphi_1, \varphi_2, \ldots, \varphi_k \vdash \psi$ for $\Gamma \vdash \psi$, in particular $\varphi \vdash \psi$ means that $\{\varphi\} \vdash \psi$.

Note that if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \ldots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i, then $\Gamma \vdash \psi$.

fol:ntd:ptn: prop:incons

Proposition ntd.16. *The following are equivalent.*

1. *Γ is inconsistent.*
2. *$\Gamma \vdash \varphi$ for every sentence φ.*
3. *$\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ for some sentence φ.*

Proof. Exercise. □

Problem ntd.6. Prove Proposition ntd.16

Proposition ntd.17 (Compactness). fol:ntd:ptn: prop:proves-compact

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*
2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof. 1. If $\Gamma \vdash \varphi$, then there is a derivation δ of φ from Γ. Let Γ_0 be the set of undischarged assumptions of δ. Since any derivation is finite, Γ_0 can only contain finitely many sentences. So, δ is a derivation of φ from a finite $\Gamma_0 \subseteq \Gamma$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \bot$. □

ntd.8 Derivability and Consistency

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem. fol:ntd:prv: sec

Proposition ntd.18. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.* fol:ntd:prv: prop:provability-contr

Proof. Let the derivation of φ from Γ be δ_1 and the derivation of $\bot$ from $\Gamma \cup \{\varphi\}$ be δ_2. We can then derive:

$$\dfrac{{}^{1}\dfrac{\begin{array}{c}\Gamma, [\varphi]^1\\ \vdots\ \delta_2\\ \bot\end{array}}{\neg\varphi}\ \neg\text{Intro} \qquad \begin{array}{c}\Gamma\\ \vdots\ \delta_1\\ \varphi\end{array}}{\bot}\ \neg\text{Elim}$$

In the new derivation, the assumption φ is discharged, so it is a derivation from Γ. □

Proposition ntd.19. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.* fol:ntd:prv: prop:prov-incons

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation δ_0 of φ from undischarged assumptions Γ. We obtain a derivation of $\bot$ from $\Gamma \cup \{\neg\varphi\}$ as follows:

$$\dfrac{\neg\varphi \qquad \begin{array}{c}\Gamma\\ \vdots\ \delta_0\\ \varphi\end{array}}{\bot}\ \neg\text{Elim}$$

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, and let δ_1 be the corresponding derivation of $\bot$ from undischarged assumptions in $\Gamma \cup \{\neg\varphi\}$. We obtain a derivation of φ from Γ alone by using $\bot_C$:

$$
\begin{array}{c}
\Gamma, [\neg\varphi]^1 \\
\vdots\ \delta_1 \\
{}^{1}\ \dfrac{\bot}{\varphi}\ \bot_C
\end{array}
$$

□

Problem ntd.7. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:ntd:prv: prop:explicit-inc

Proposition ntd.20. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a derivation δ of φ from Γ. Consider this simple application of the $\neg$Elim rule:

$$
\dfrac{\neg\varphi \qquad \begin{array}{c}\Gamma \\ \vdots\ \delta \\ \varphi\end{array}}{\bot}\ \neg\text{Elim}
$$

Since $\neg\varphi \in \Gamma$, all undischarged assumptions are in Γ, this shows that $\Gamma \vdash \bot$. □

fol:ntd:prv: prop:provability-exhaustive

Proposition ntd.21. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.*

Proof. There are derivations δ_1 and δ_2 of $\bot$ from $\Gamma \cup \{\varphi\}$ and $\bot$ from $\Gamma \cup \{\neg\varphi\}$, respectively. We can then derive

$$
\dfrac{
\begin{array}{c}\Gamma, [\neg\varphi]^2 \\ \vdots\ \delta_2 \\ {}^{2}\ \dfrac{\bot}{\neg\neg\varphi}\ \neg\text{Intro}\end{array}
\qquad
\begin{array}{c}\Gamma, [\varphi]^1 \\ \vdots\ \delta_1 \\ {}^{1}\ \dfrac{\bot}{\neg\varphi}\ \neg\text{Intro}\end{array}
}{\bot}\ \neg\text{Elim}
$$

Since the assumptions φ and $\neg\varphi$ are discharged, this is a derivation of $\bot$ from Γ alone. Hence Γ is inconsistent. □

ntd.9 Derivability and the Propositional Connectives

explanation We establish that the derivability relation $\vdash$ of natural deduction is strong enough to establish some basic facts involving the propositional connectives, such as that $\varphi \land \psi \vdash \varphi$ and $\varphi, \varphi \to \psi \vdash \psi$ (modus ponens). These facts are needed for the proof of the completeness theorem. fol:ntd:ppr:sec

Proposition ntd.22. fol:ntd:ppr:prop:provability-land

1. *Both* $\varphi \land \psi \vdash \varphi$ *and* $\varphi \land \psi \vdash \psi$ fol:ntd:ppr:prop:provability-land-left
2. $\varphi, \psi \vdash \varphi \land \psi$. fol:ntd:ppr:prop:provability-land-right

Proof. 1. We can derive both

$$\frac{\varphi \land \psi}{\varphi}\,\land\text{Elim} \qquad \frac{\varphi \land \psi}{\psi}\,\land\text{Elim}$$

2. We can derive:

$$\frac{\varphi \qquad \psi}{\varphi \land \psi}\,\land\text{Intro}$$

□

Proposition ntd.23. fol:ntd:ppr:prop:provability-lor

1. $\varphi \lor \psi, \neg\varphi, \neg\psi$ *is inconsistent.*
2. *Both* $\varphi \vdash \varphi \lor \psi$ *and* $\psi \vdash \varphi \lor \psi$.

Proof. 1. Consider the following derivation:

$$ {}_1\,\frac{\varphi \lor \psi \qquad \dfrac{\neg\varphi \qquad [\varphi]^1}{\bot}\,\neg\text{Elim} \qquad \dfrac{\neg\psi \qquad [\psi]^1}{\bot}\,\neg\text{Elim}}{\bot}\,\lor\text{Elim}$$

This is a derivation of $\bot$ from undischarged assumptions $\varphi \lor \psi$, $\neg\varphi$, and $\neg\psi$.

2. We can derive both

$$\frac{\varphi}{\varphi \lor \psi}\,\lor\text{Intro} \qquad \frac{\psi}{\varphi \lor \psi}\,\lor\text{Intro}$$

□

Proposition ntd.24. fol:ntd:ppr:prop:provability-lif

1. $\varphi, \varphi \to \psi \vdash \psi$. fol:ntd:ppr:prop:provability-lif-left
2. *Both* $\neg\varphi \vdash \varphi \to \psi$ *and* $\psi \vdash \varphi \to \psi$. fol:ntd:ppr:prop:provability-lif-right

Proof. 1. We can derive:

$$\dfrac{\varphi \to \psi \qquad \varphi}{\psi} \to\text{Elim}$$

2. This is shown by the following two derivations:

$$\dfrac{\dfrac{\dfrac{\neg\varphi \qquad [\varphi]^1}{\bot}\neg\text{Elim}}{\psi}\bot_I}{\varphi \to \psi}{\to}\text{Intro}^{1} \qquad \dfrac{\psi}{\varphi \to \psi}{\to}\text{Intro}$$

Note that →Intro may, but does not have to, discharge the assumption φ. □

ntd.10 Derivability and the Quantifiers

fol:ntd:qpr: sec

The completeness theorem also requires that the natural deduction rules yield the facts about $\vdash$ established in this section. explanation

fol:ntd:qpr: thm:strong-generalization

Theorem ntd.25. *If c is a constant not occurring in Γ or $\varphi(x)$ and $\Gamma \vdash \varphi(c)$, then $\Gamma \vdash \forall x\, \varphi(x)$.*

Proof. Let δ be a derivation of $\varphi(c)$ from Γ. By adding a ∀Intro inference, we obtain a derivation of $\forall x\, \varphi(x)$. Since c does not occur in Γ or $\varphi(x)$, the eigenvariable condition is satisfied. □

fol:ntd:qpr: prop:provability-quantifiers

Proposition ntd.26.

1. $\varphi(t) \vdash \exists x\, \varphi(x)$.
2. $\forall x\, \varphi(x) \vdash \varphi(t)$.

Proof. 1. The following is a derivation of $\exists x\, \varphi(x)$ from $\varphi(t)$:

$$\dfrac{\varphi(t)}{\exists x\, \varphi(x)} \exists\text{Intro}$$

2. The following is a derivation of $\varphi(t)$ from $\forall x\, \varphi(x)$:

$$\dfrac{\forall x\, \varphi(x)}{\varphi(t)} \forall\text{Elim}$$

□

ntd.11 Soundness

explanation A derivation system, such as natural deduction, is *sound* if it cannot derive things that do not actually follow. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that fol:ntd:sou:sec

1. every derivable sentence is valid;
2. if a sentence is derivable from some others, it is also a consequence of them;
3. if a set of sentences is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

Theorem ntd.27 (Soundness). *If φ is derivable from the undischarged assumptions Γ, then $\Gamma \vDash \varphi$.* *fol:ntd:sou:thm:soundness*

Proof. Let δ be a derivation of φ. We proceed by induction on the number of inferences in δ.

For the induction basis we show the claim if the number of inferences is 0. In this case, δ consists only of a single sentence φ, i.e., an assumption. That assumption is undischarged, since assumptions can only be discharged by inferences, and there are no inferences. So, any structure $\mathfrak{M}$ that satisfies all of the undischarged assumptions of the proof also satisfies φ.

Now for the inductive step. Suppose that δ contains n inferences. The premise(s) of the lowermost inference are derived using sub-derivations, each of which contains fewer than n inferences. We assume the induction hypothesis: The premises of the lowermost inference follow from the undischarged assumptions of the sub-derivations ending in those premises. We have to show that the conclusion φ follows from the undischarged assumptions of the entire proof.

We distinguish cases according to the type of the lowermost inference. First, we consider the possible inferences with only one premise.

1. Suppose that the last inference is $\neg$Intro: The derivation has the form

$$\cfrac{\begin{array}{c}\Gamma, [\varphi]^n \\ \vdots\ \delta_1 \\ \bot\end{array}}{\neg\varphi}\ {\neg}\mathrm{Intro}\ (n)$$

By inductive hypothesis, $\bot$ follows from the undischarged assumptions $\Gamma \cup \{\varphi\}$ of δ_1. Consider a structure $\mathfrak{M}$. We need to show that, if $\mathfrak{M} \vDash \Gamma$, then $\mathfrak{M} \vDash \neg\varphi$. Suppose for reductio that $\mathfrak{M} \vDash \Gamma$, but $\mathfrak{M} \nvDash \neg\varphi$, i.e., $\mathfrak{M} \vDash \varphi$. This would mean that $\mathfrak{M} \vDash \Gamma \cup \{\varphi\}$. This is contrary to our inductive hypothesis. So, $\mathfrak{M} \vDash \neg\varphi$.

2. The last inference is $\wedge$Elim: There are two variants: φ or ψ may be inferred from the premise $\varphi \wedge \psi$. Consider the first case. The derivation δ looks like this:

$$\dfrac{\begin{matrix}\Gamma \\ \vdots\,\delta_1 \\ \varphi \wedge \psi\end{matrix}}{\varphi}\ \wedge\text{Elim}$$

By inductive hypothesis, $\varphi \wedge \psi$ follows from the undischarged assumptions Γ of δ_1. Consider a structure $\mathfrak{M}$. We need to show that, if $\mathfrak{M} \vDash \Gamma$, then $\mathfrak{M} \vDash \varphi$. Suppose $\mathfrak{M} \vDash \Gamma$. By our inductive hypothesis ($\Gamma \vDash \varphi \wedge \psi$), we know that $\mathfrak{M} \vDash \varphi \wedge \psi$. By definition, $\mathfrak{M} \vDash \varphi \wedge \psi$ iff $\mathfrak{M} \vDash \varphi$ and $\mathfrak{M} \vDash \psi$. (The case where ψ is inferred from $\varphi \wedge \psi$ is handled similarly.)

3. The last inference is $\vee$Intro: There are two variants: $\varphi \vee \psi$ may be inferred from the premise φ or the premise ψ. Consider the first case. The derivation has the form

$$\dfrac{\begin{matrix}\Gamma \\ \vdots\,\delta_1 \\ \varphi\end{matrix}}{\varphi \vee \psi}\ \vee\text{Intro}$$

By inductive hypothesis, φ follows from the undischarged assumptions Γ of δ_1. Consider a structure $\mathfrak{M}$. We need to show that, if $\mathfrak{M} \vDash \Gamma$, then $\mathfrak{M} \vDash \varphi \vee \psi$. Suppose $\mathfrak{M} \vDash \Gamma$; then $\mathfrak{M} \vDash \varphi$ since $\Gamma \vDash \varphi$ (the inductive hypothesis). So it must also be the case that $\mathfrak{M} \vDash \varphi \vee \psi$. (The case where $\varphi \vee \psi$ is inferred from ψ is handled similarly.)

4. The last inference is $\to$Intro: $\varphi \to \psi$ is inferred from a subproof with assumption φ and conclusion ψ, i.e.,

$$n\ \dfrac{\begin{matrix}\Gamma, [\varphi]^n \\ \vdots\,\delta_1 \\ \psi\end{matrix}}{\varphi \to \psi}\ \to\text{Intro}$$

By inductive hypothesis, ψ follows from the undischarged assumptions of δ_1, i.e., $\Gamma \cup \{\varphi\} \vDash \psi$. Consider a structure $\mathfrak{M}$. The undischarged assumptions of δ are just Γ, since φ is discharged at the last inference. So we need to show that $\Gamma \vDash \varphi \rightarrow \psi$. For reductio, suppose that for some structure $\mathfrak{M}$, $\mathfrak{M} \vDash \Gamma$ but $\mathfrak{M} \nvDash \varphi \rightarrow \psi$. So, $\mathfrak{M} \vDash \varphi$ and $\mathfrak{M} \nvDash \psi$. But by hypothesis, ψ is a consequence of $\Gamma \cup \{\varphi\}$, i.e., $\mathfrak{M} \vDash \psi$, which is a contradiction. So, $\Gamma \vDash \varphi \rightarrow \psi$.

5. The last inference is $\bot_I$: Here, δ ends in

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\, \delta_1\\ \bot\end{array}}{\varphi}\ \bot_I$$

By induction hypothesis, $\Gamma \vDash \bot$. We have to show that $\Gamma \vDash \varphi$. Suppose not; then for some $\mathfrak{M}$ we have $\mathfrak{M} \vDash \Gamma$ and $\mathfrak{M} \nvDash \varphi$. But we always have $\mathfrak{M} \nvDash \bot$, so this would mean that $\Gamma \nvDash \bot$, contrary to the induction hypothesis.

6. The last inference is $\bot_C$: Exercise.

7. The last inference is $\forall$Intro: Then δ has the form

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\, \delta_1\\ \varphi(a)\end{array}}{\forall x\, \varphi(x)}\ \forall\text{Intro}$$

The premise $\varphi(a)$ is a consequence of the undischarged assumptions Γ by induction hypothesis. Consider some structure, $\mathfrak{M}$, such that $\mathfrak{M} \vDash \Gamma$. We need to show that $\mathfrak{M} \vDash \forall x\, \varphi(x)$. Since $\forall x\, \varphi(x)$ is a sentence, this means we have to show that for every variable assignment s, $\mathfrak{M}, s \vDash \varphi(x)$ (**??**). Since Γ consists entirely of sentences, $\mathfrak{M}, s \vDash \psi$ for all $\psi \in \Gamma$ by **??**. Let $\mathfrak{M}'$ be like $\mathfrak{M}$ except that $a^{\mathfrak{M}'} = s(x)$. Since a does not occur in Γ, $\mathfrak{M}' \vDash \Gamma$ by **??**. Since $\Gamma \vDash \varphi(a)$, $\mathfrak{M}' \vDash \varphi(a)$. Since $\varphi(a)$ is a sentence, $\mathfrak{M}', s \vDash \varphi(a)$ by **??**. $\mathfrak{M}', s \vDash \varphi(x)$ iff $\mathfrak{M}' \vDash \varphi(a)$ by **??** (recall that $\varphi(a)$ is just $\varphi(x)[a/x]$). So, $\mathfrak{M}', s \vDash \varphi(x)$. Since a does not occur in $\varphi(x)$, by **??**, $\mathfrak{M}, s \vDash \varphi(x)$. But s was an arbitrary variable assignment, so $\mathfrak{M} \vDash \forall x\, \varphi(x)$.

8. The last inference is $\exists$Intro: Exercise.

9. The last inference is $\forall$Elim: Exercise.

Now let's consider the possible inferences with several premises: $\lor$Elim, $\land$Intro, $\rightarrow$Elim, and $\exists$Elim.

1. The last inference is $\land$Intro. $\varphi \land \psi$ is inferred from the premises φ and ψ and δ has the form

$$\dfrac{\begin{matrix}\Gamma_1 \\ \vdots\ \delta_1 \\ \varphi\end{matrix} \qquad \begin{matrix}\Gamma_2 \\ \vdots\ \delta_2 \\ \psi\end{matrix}}{\varphi \land \psi} \land\text{Intro}$$

   By induction hypothesis, φ follows from the undischarged assumptions Γ_1 of δ_1 and ψ follows from the undischarged assumptions Γ_2 of δ_2. The undischarged assumptions of δ are $\Gamma_1 \cup \Gamma_2$, so we have to show that $\Gamma_1 \cup \Gamma_2 \vDash \varphi \land \psi$. Consider a structure $\mathfrak{M}$ with $\mathfrak{M} \vDash \Gamma_1 \cup \Gamma_2$. Since $\mathfrak{M} \vDash \Gamma_1$, it must be the case that $\mathfrak{M} \vDash \varphi$ as $\Gamma_1 \vDash \varphi$, and since $\mathfrak{M} \vDash \Gamma_2$, $\mathfrak{M} \vDash \psi$ since $\Gamma_2 \vDash \psi$. Together, $\mathfrak{M} \vDash \varphi \land \psi$.

2. The last inference is $\lor$Elim: Exercise.

3. The last inference is $\rightarrow$Elim. ψ is inferred from the premises $\varphi \rightarrow \psi$ and φ. The derivation δ looks like this:

$$\dfrac{\begin{matrix}\Gamma_1 \\ \vdots\ \delta_1 \\ \varphi \rightarrow \psi\end{matrix} \qquad \begin{matrix}\Gamma_2 \\ \vdots\ \delta_2 \\ \varphi\end{matrix}}{\psi} \rightarrow\text{Elim}$$

   By induction hypothesis, $\varphi \rightarrow \psi$ follows from the undischarged assumptions Γ_1 of δ_1 and φ follows from the undischarged assumptions Γ_2 of δ_2. Consider a structure $\mathfrak{M}$. We need to show that, if $\mathfrak{M} \vDash \Gamma_1 \cup \Gamma_2$, then $\mathfrak{M} \vDash \psi$. Suppose $\mathfrak{M} \vDash \Gamma_1 \cup \Gamma_2$. Since $\Gamma_1 \vDash \varphi \rightarrow \psi$, $\mathfrak{M} \vDash \varphi \rightarrow \psi$. Since $\Gamma_2 \vDash \varphi$, we have $\mathfrak{M} \vDash \varphi$. This means that $\mathfrak{M} \vDash \psi$ (For if $\mathfrak{M} \nvDash \psi$, since $\mathfrak{M} \vDash \varphi$, we'd have $\mathfrak{M} \nvDash \varphi \rightarrow \psi$, contradicting $\mathfrak{M} \vDash \varphi \rightarrow \psi$).

4. The last inference is $\lnot$Elim: Exercise.

5. The last inference is $\exists$Elim: Exercise. □

Problem ntd.8. Complete the proof of Theorem ntd.27.

fol:ntd:sou: cor:weak-soundness **Corollary ntd.28.** *If $\vdash \varphi$, then φ is valid.*

fol:ntd:sou: cor:consistency-soundness **Corollary ntd.29.** *If Γ is satisfiable, then it is consistent.*

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then $\Gamma \vdash \bot$, i.e., there is a derivation of $\bot$ from undischarged assumptions in Γ. By Theorem ntd.27, any structure $\mathfrak{M}$ that satisfies Γ must satisfy $\bot$. Since $\mathfrak{M} \nvDash \bot$ for every structure $\mathfrak{M}$, no $\mathfrak{M}$ can satisfy Γ, i.e., Γ is not satisfiable. □

ntd.12 Derivations with Identity predicate

Derivations with identity predicate require additional inference rules. fol:ntd:ide:sec

$$\frac{}{t = t}\ {=}\mathrm{Intro} \qquad\qquad \frac{t_1 = t_2 \qquad \varphi(t_1)}{\varphi(t_2)}\ {=}\mathrm{Elim} \qquad \frac{t_1 = t_2 \qquad \varphi(t_2)}{\varphi(t_1)}\ {=}\mathrm{Elim}$$

In the above rules, t, t_1, and t_2 are closed terms. The =Intro rule allows us to derive any identity statement of the form $t = t$ outright, from no assumptions.

Example ntd.30. If s and t are closed terms, then $\varphi(s), s = t \vdash \varphi(t)$:

$$\frac{s = t \qquad \varphi(s)}{\varphi(t)}\ {=}\mathrm{Elim}$$

This may be familiar as the “principle of substitutability of identicals,” or Leibniz’ Law.

Problem ntd.9. Prove that = is both symmetric and transitive, i.e., give derivations of $\forall x\, \forall y\, (x = y \rightarrow y = x)$ and $\forall x\, \forall y\, \forall z((x = y \land y = z) \rightarrow x = z)$

Example ntd.31. We derive the sentence

$$\forall x\, \forall y\, ((\varphi(x) \land \varphi(y)) \rightarrow x = y)$$

from the sentence

$$\exists x\, \forall y\, (\varphi(y) \rightarrow y = x)$$

We develop the derivation backwards:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\exists x\, \forall y\, (\varphi(y) \rightarrow y = x) \quad [\varphi(a) \land \varphi(b)]^1 \\ \vdots \\ a = b\end{array}}{((\varphi(a) \land \varphi(b)) \rightarrow a = b)}\ {}^{1}\rightarrow\mathrm{Intro}}{\forall y\, ((\varphi(a) \land \varphi(y)) \rightarrow a = y)}\ \forall\mathrm{Intro}}{\forall x\, \forall y\, ((\varphi(x) \land \varphi(y)) \rightarrow x = y)}\ \forall\mathrm{Intro}$$

We'll now have to use the main assumption: since it is an existential formula, we use ∃Elim to derive the intermediary conclusion $a = b$.

$$
\dfrac{\dfrac{\dfrac{{}_2\;\dfrac{\exists x\, \forall y\, (\varphi(y) \to y = x) \qquad \begin{array}{c}[\forall y\, (\varphi(y) \to y = c)]^2 \\ [\varphi(a) \land \varphi(b)]^1 \\ \vdots \\ a = b\end{array}}{a = b}\;\exists\text{Elim}}{((\varphi(a) \land \varphi(b)) \to a = b)}\;{}_1\,{\to}\text{Intro}}{\forall y\, ((\varphi(a) \land \varphi(y)) \to a = y)}\;\forall\text{Intro}}{\forall x\, \forall y\, ((\varphi(x) \land \varphi(y)) \to x = y)}\;\forall\text{Intro}
$$

The sub-derivation on the top right is completed by using its assumptions to show that $a = c$ and $b = c$. This requires two separate derivations. The derivation for $a = c$ is as follows:

$$
\dfrac{\dfrac{[\forall y\, (\varphi(y) \to y = c)]^2}{\varphi(a) \to a = c}\;\forall\text{Elim} \qquad \dfrac{[\varphi(a) \land \varphi(b)]^1}{\varphi(a)}\;\land\text{Elim}}{a = c}\;{\to}\text{Elim}
$$

From $a = c$ and $b = c$ we derive $a = b$ by =Elim.

Problem ntd.10. Give derivations of the following formulas:

1. $\forall x\, \forall y\, ((x = y \land \varphi(x)) \to \varphi(y))$
2. $\exists x\, \varphi(x) \land \forall y\, \forall z\, ((\varphi(y) \land \varphi(z)) \to y = z) \to \exists x\, (\varphi(x) \land \forall y\, (\varphi(y) \to y = x))$

ntd.13 Soundness with Identity predicate

fol:ntd:sid:sec

Proposition ntd.32. *Natural deduction with rules for = is sound.*

Proof. Any formula of the form $t = t$ is valid, since for every structure $\mathfrak{M}$, $\mathfrak{M} \vDash t = t$. (Note that we assume the term t to be closed, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a derivation is =Elim, i.e., the derivation has the following form:

$$
\dfrac{\begin{array}{c}\Gamma_1 \\ \vdots\,\delta_1 \\ t_1 = t_2\end{array} \qquad \begin{array}{c}\Gamma_2 \\ \vdots\,\delta_2 \\ \varphi(t_1)\end{array}}{\varphi(t_2)}\;{=}\text{Elim}
$$

The premises $t_1 = t_2$ and $\varphi(t_1)$ are derived from undischarged assumptions Γ_1 and Γ_2, respectively. We want to show that $\varphi(t_2)$ follows from $\Gamma_1 \cup \Gamma_2$. Consider a structure $\mathfrak{M}$ with $\mathfrak{M} \vDash \Gamma_1 \cup \Gamma_2$. By induction hypothesis, $\mathfrak{M} \vDash \varphi(t_1)$ and $\mathfrak{M} \vDash t_1 = t_2$. Therefore, $\mathrm{Val}^{\mathfrak{M}}(t_1) = \mathrm{Val}^{\mathfrak{M}}(t_2)$. Let s be any variable assignment, and $m = \mathrm{Val}^{\mathfrak{M}}(t_1) = \mathrm{Val}^{\mathfrak{M}}(t_2)$. By **??**, $\mathfrak{M}, s \vDash \varphi(t_1)$ iff $\mathfrak{M}, s[m/x] \vDash \varphi(x)$ iff $\mathfrak{M}, s \vDash \varphi(t_2)$. Since $\mathfrak{M} \vDash \varphi(t_1)$, we have $\mathfrak{M} \vDash \varphi(t_2)$. □

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Bibliography